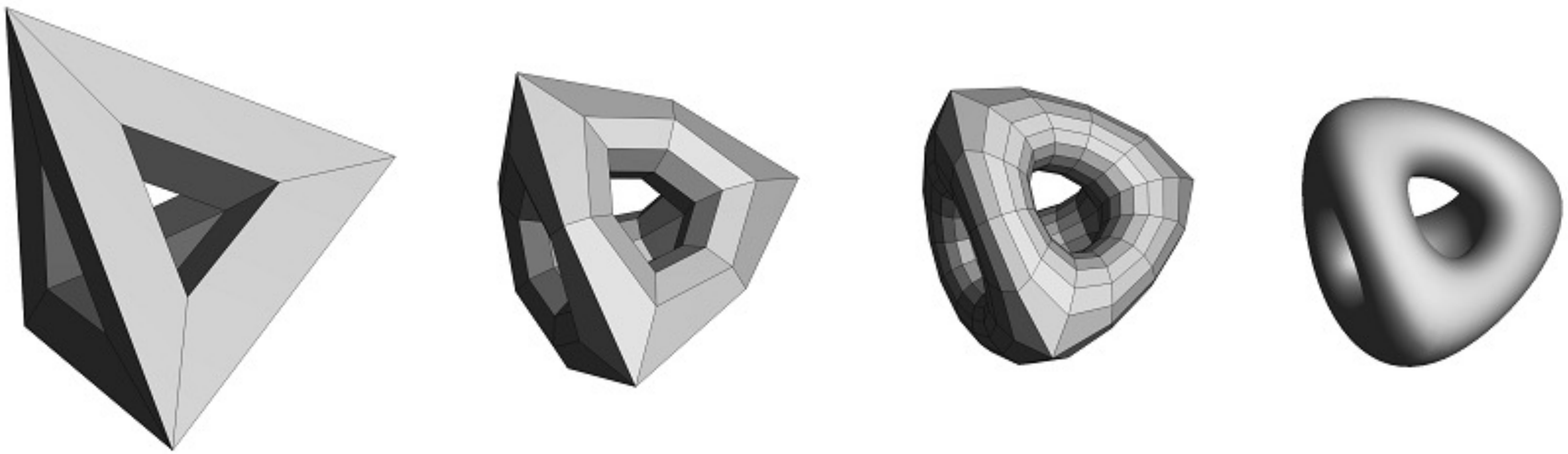
Introduction to Computer Graphics

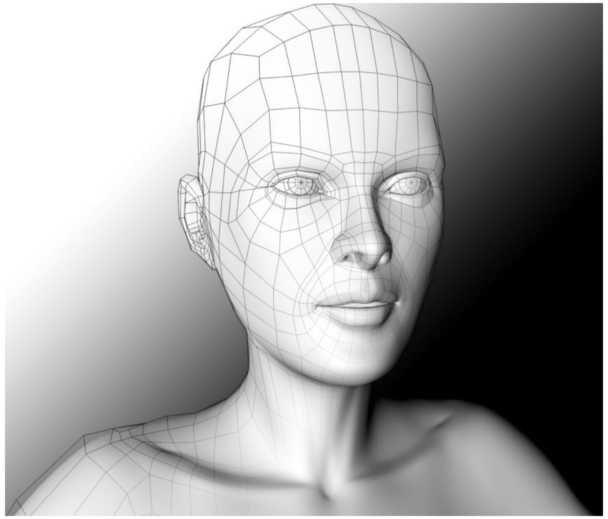
– Modeling (2) –

April 22, 2021

Kenshi Takayama

Subdivision surfaces



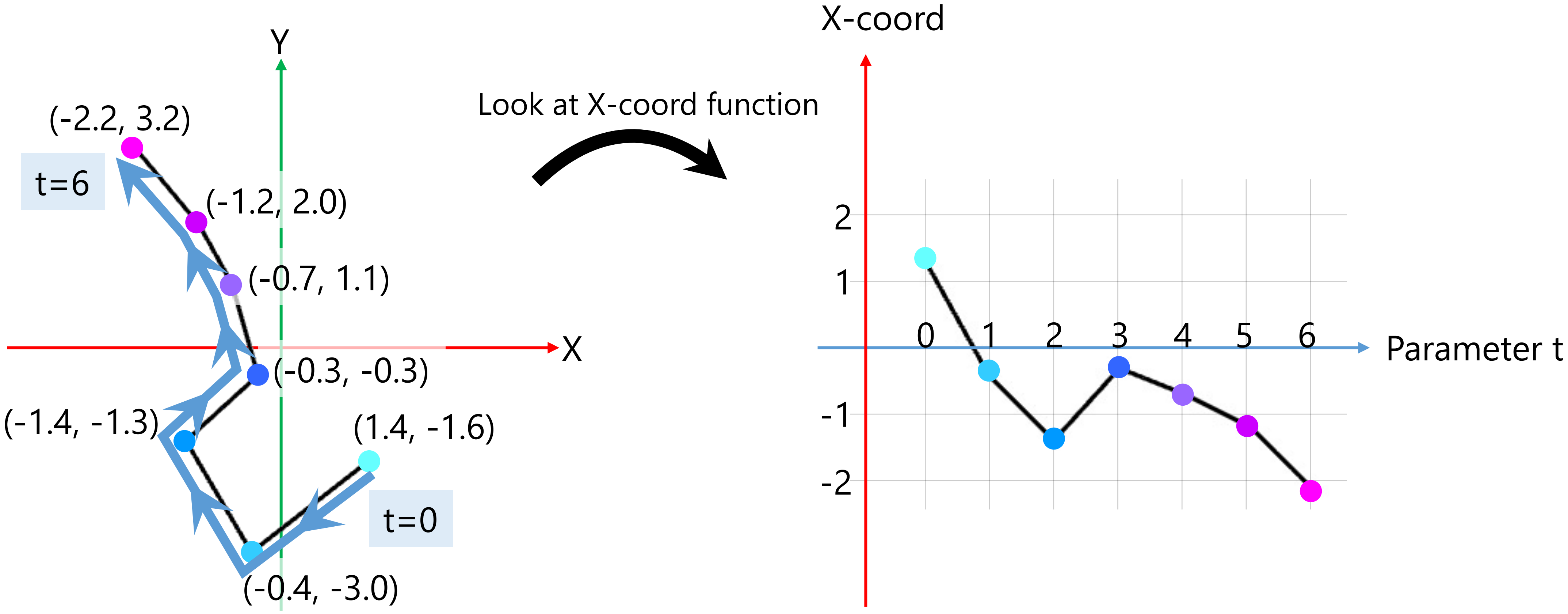


First, we'll look at its theoretical basis:

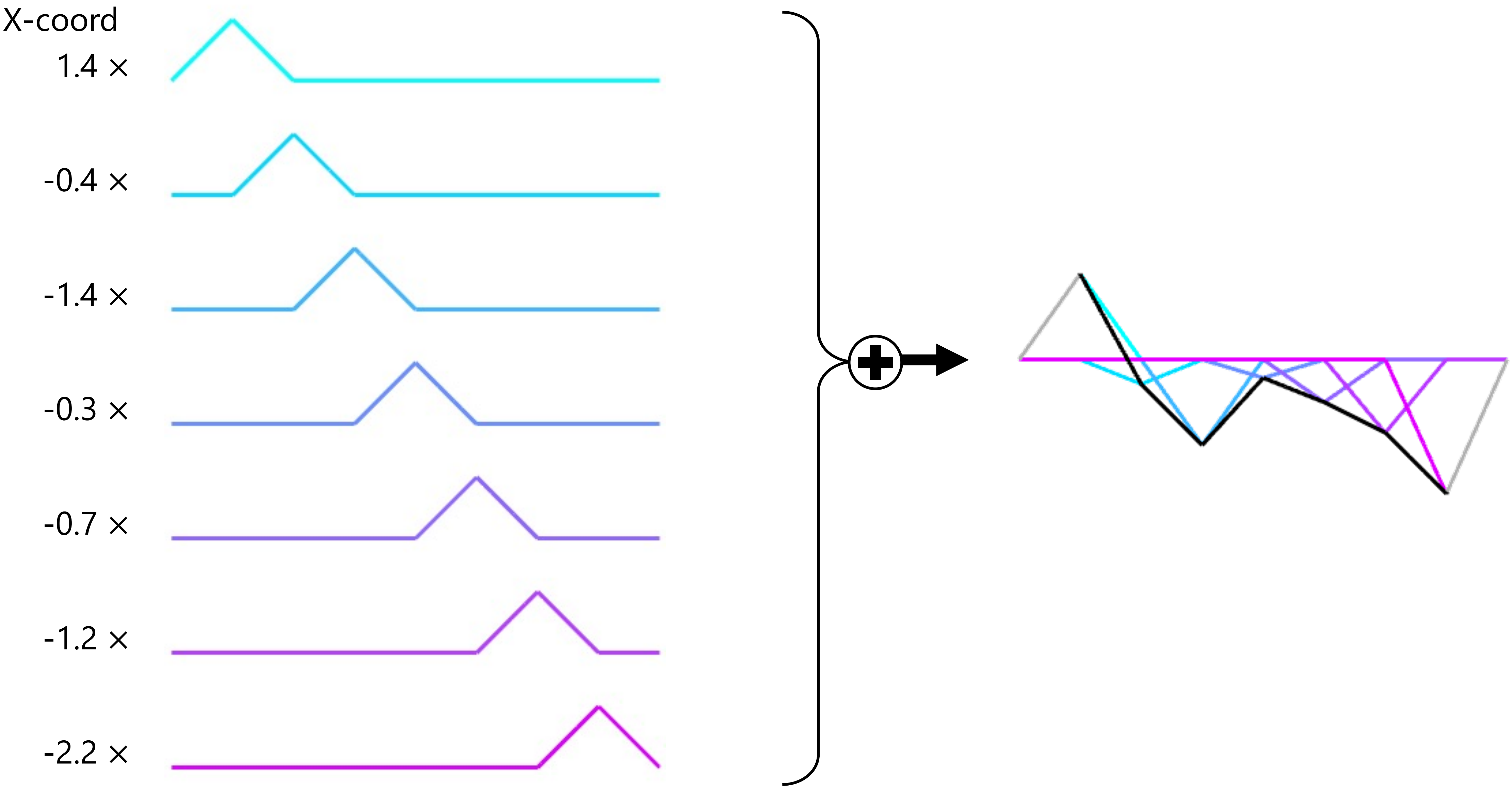
B-Spline curves

Basis

Example: 2D polyline represented as function

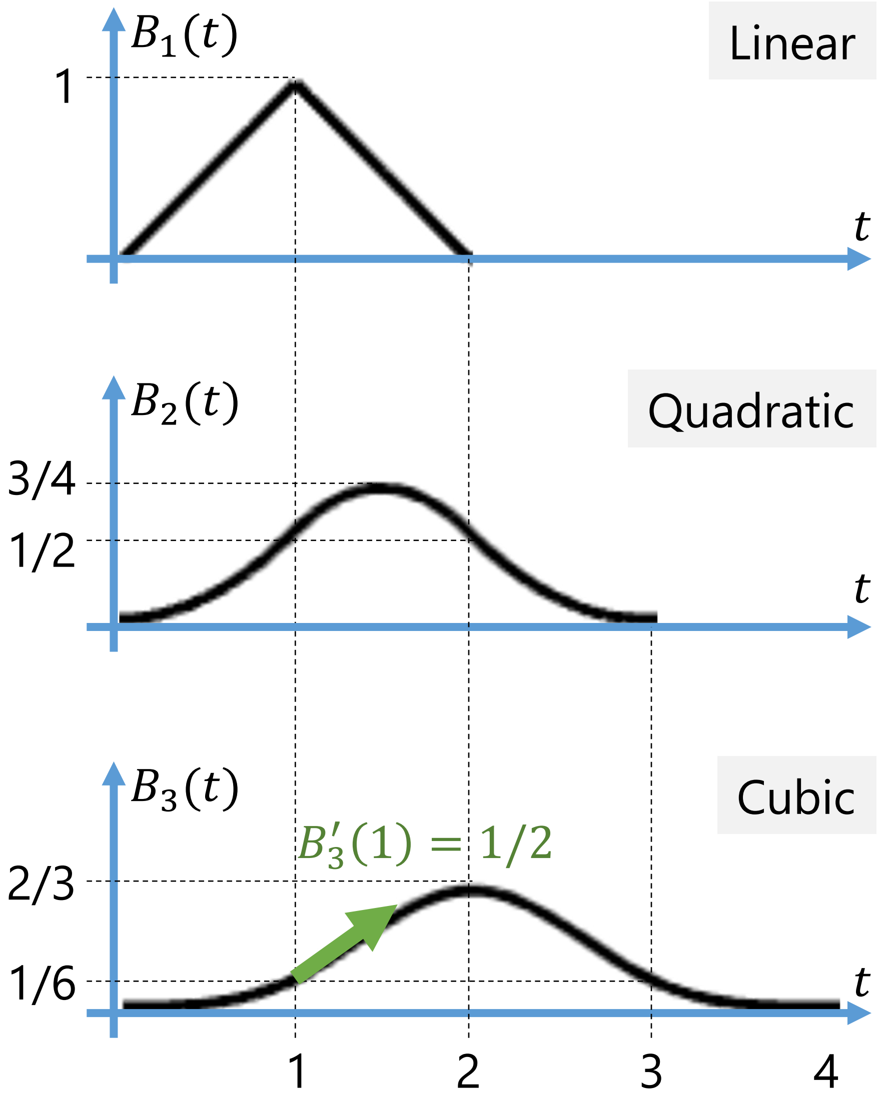


Representing polyline using linear basis

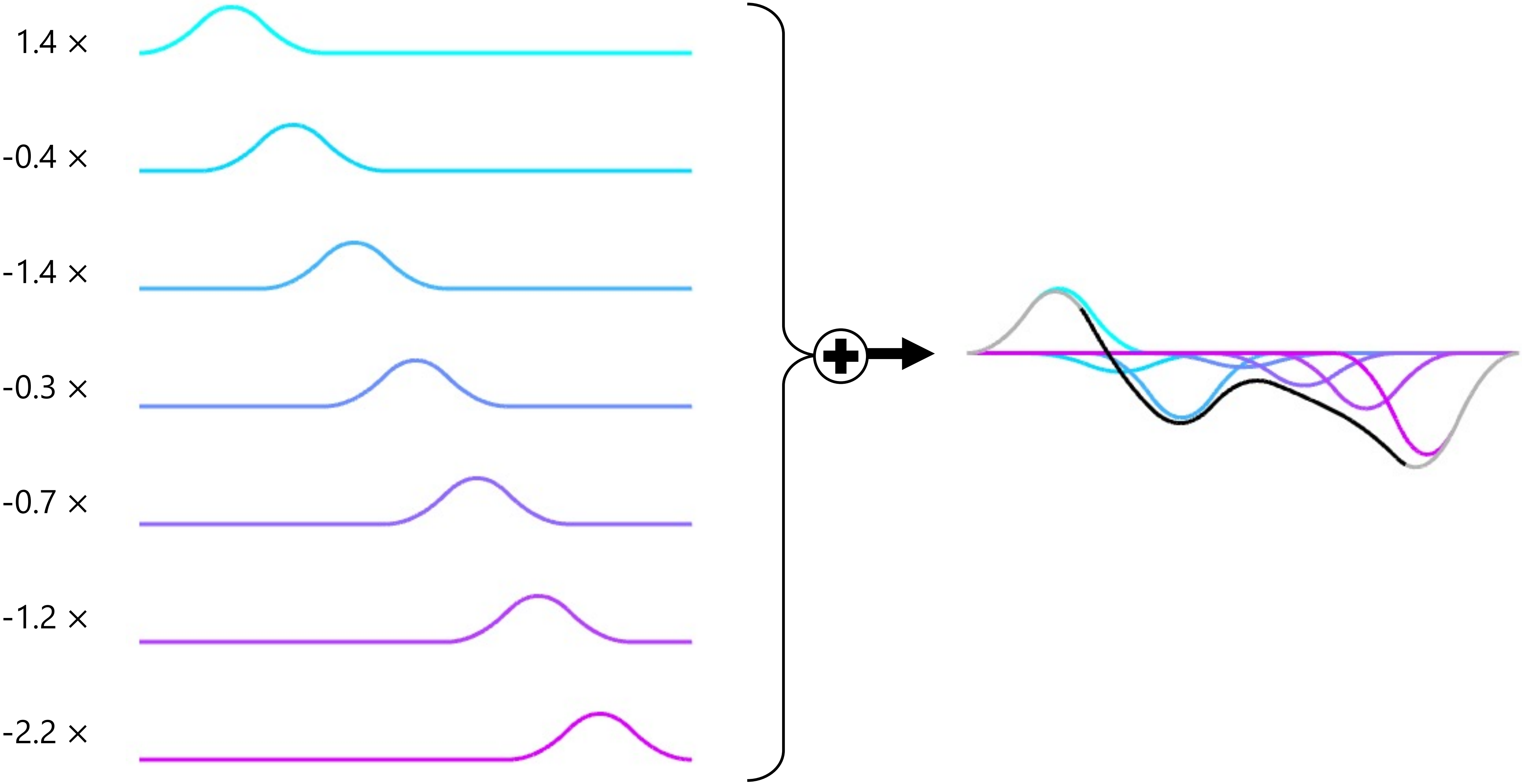


de Boor's n-th order basis

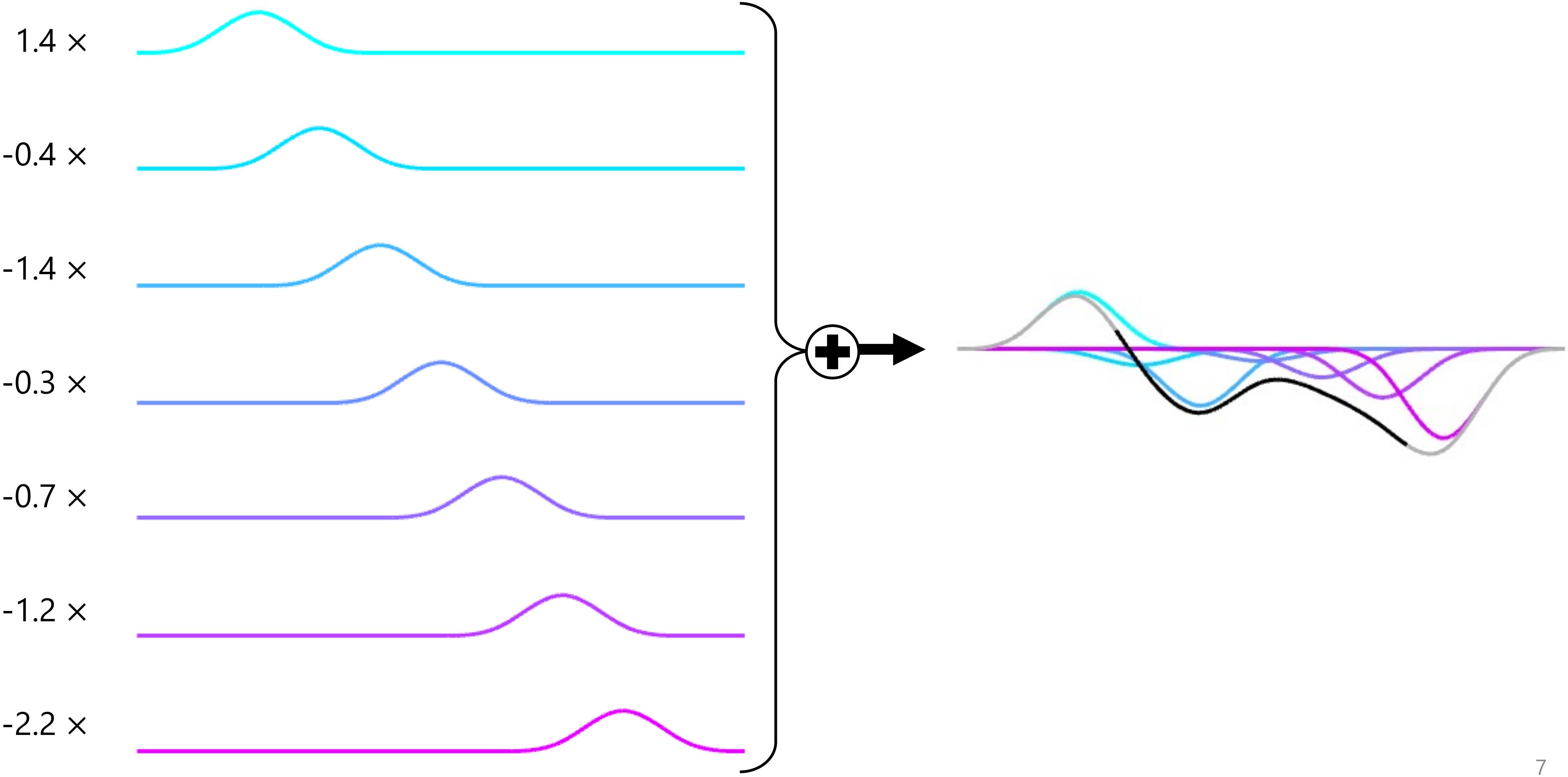
- Recursively defined:
  - $B_0(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$
  - $B_n(t) = \frac{t}{n} B_{n-1}(t) + \frac{n+1-t}{n} B_{n-1}(t-1)$
- Properties:
  - n-th order piecewise polynomial
  - Zero outside [0, n+1] (local support)
  - C^{n-1} continuous



Using quadratic basis ➔ quadratic B-spline



Using cubic basis ➔ cubic B-spline



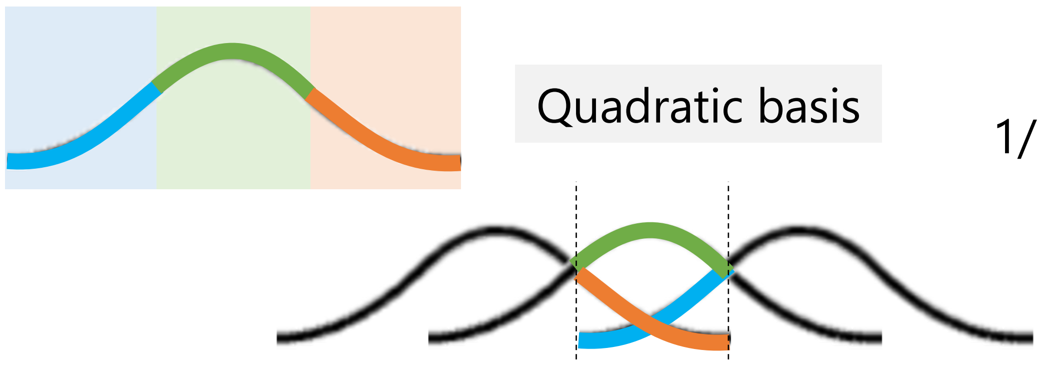
Important property of basis: partition-of-unity

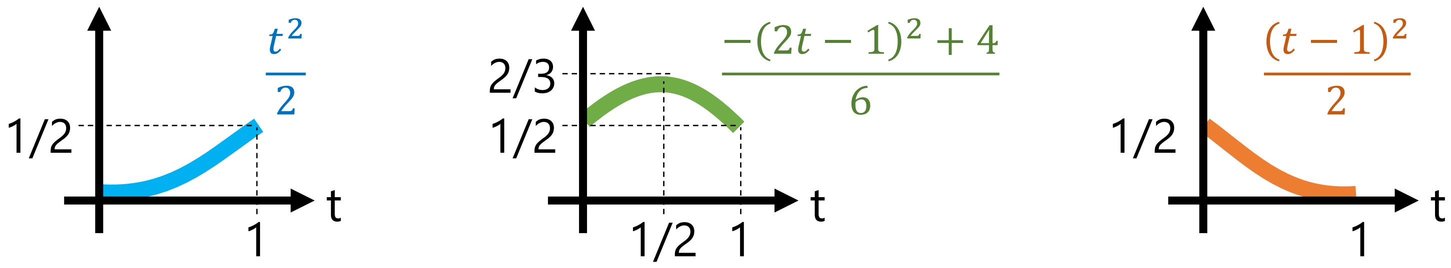
- X-coord of B-spline: $x(t) = \sum_i x_i B_n(t-i)$

- Consider moving all control points x_i by the same amount c :
  - $x(t) = \sum_i (x_i + c) B_n(t-i)$

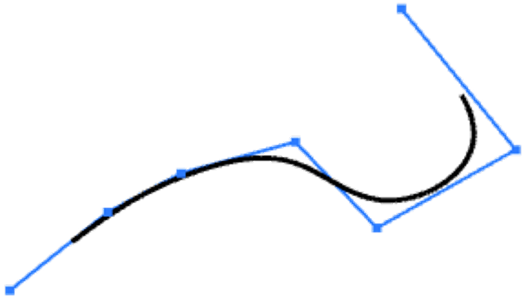
$$= \sum_i x_i B_n(t-i) + c \underbrace{\sum_i B_n(t-i)}_{1}$$

  - Partition-of-unity ensures that the entire curve is also moved by c





Cubic B-spline vs. Cubic Catmull-Rom spline

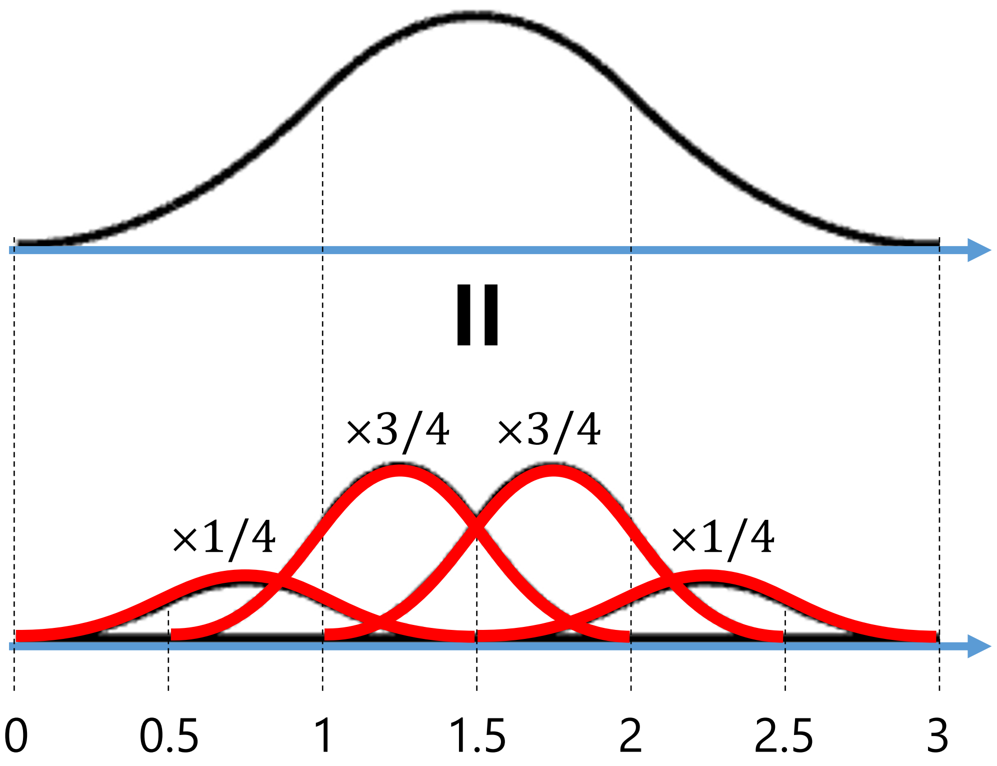
		
Representation	Piecewise cubic	Piecewise cubic
Defined as	Linear combination of cubic bases	Given coordinate value at each knot $t = t_k$, compute derivative at each knot Hermite interpolation for each interval
Passes through CPs?	No	Yes
Continuity across intervals	C^2-continuous	C^1-continuous

From B-spline to subdivision

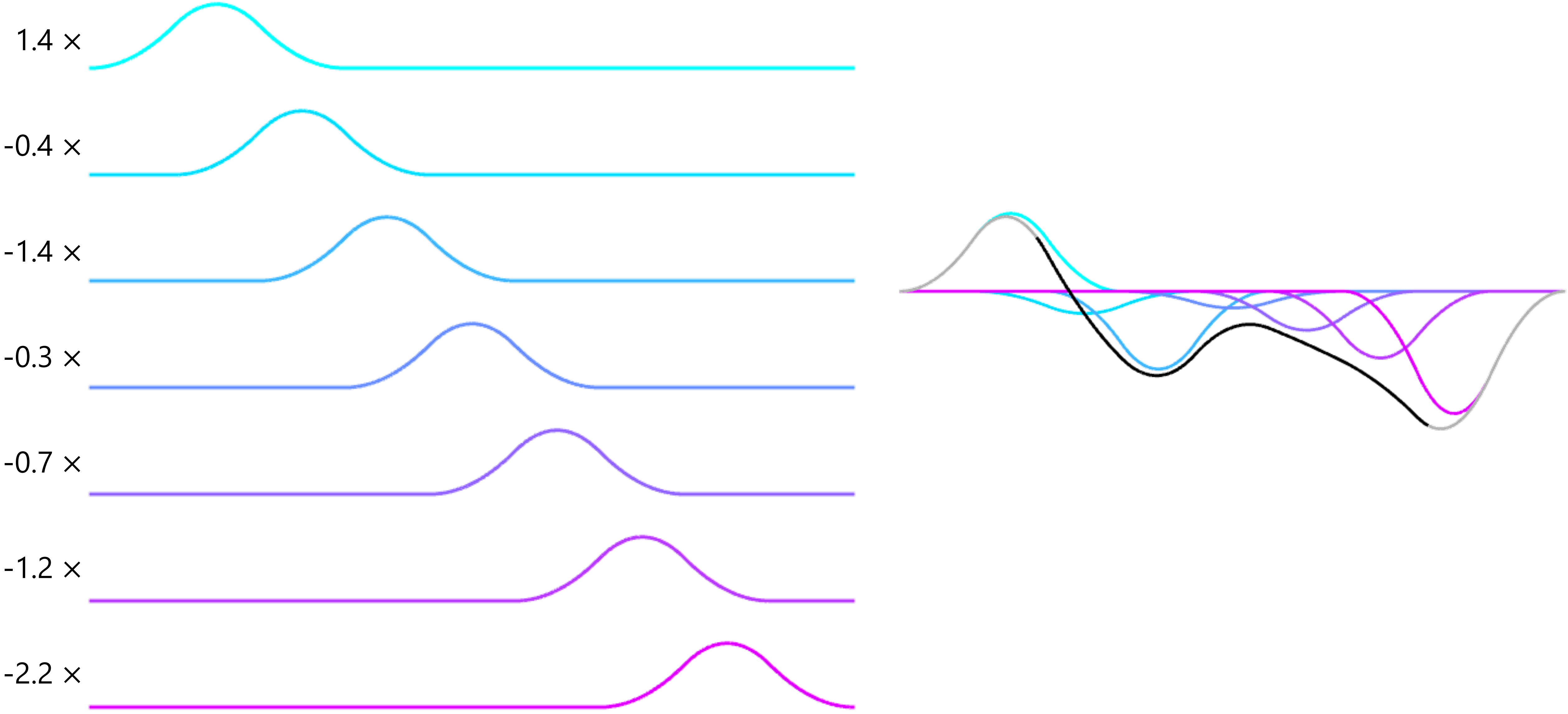
Another important property of basis function

- Can be decomposed into weighted sum of the same basis functions with halved support

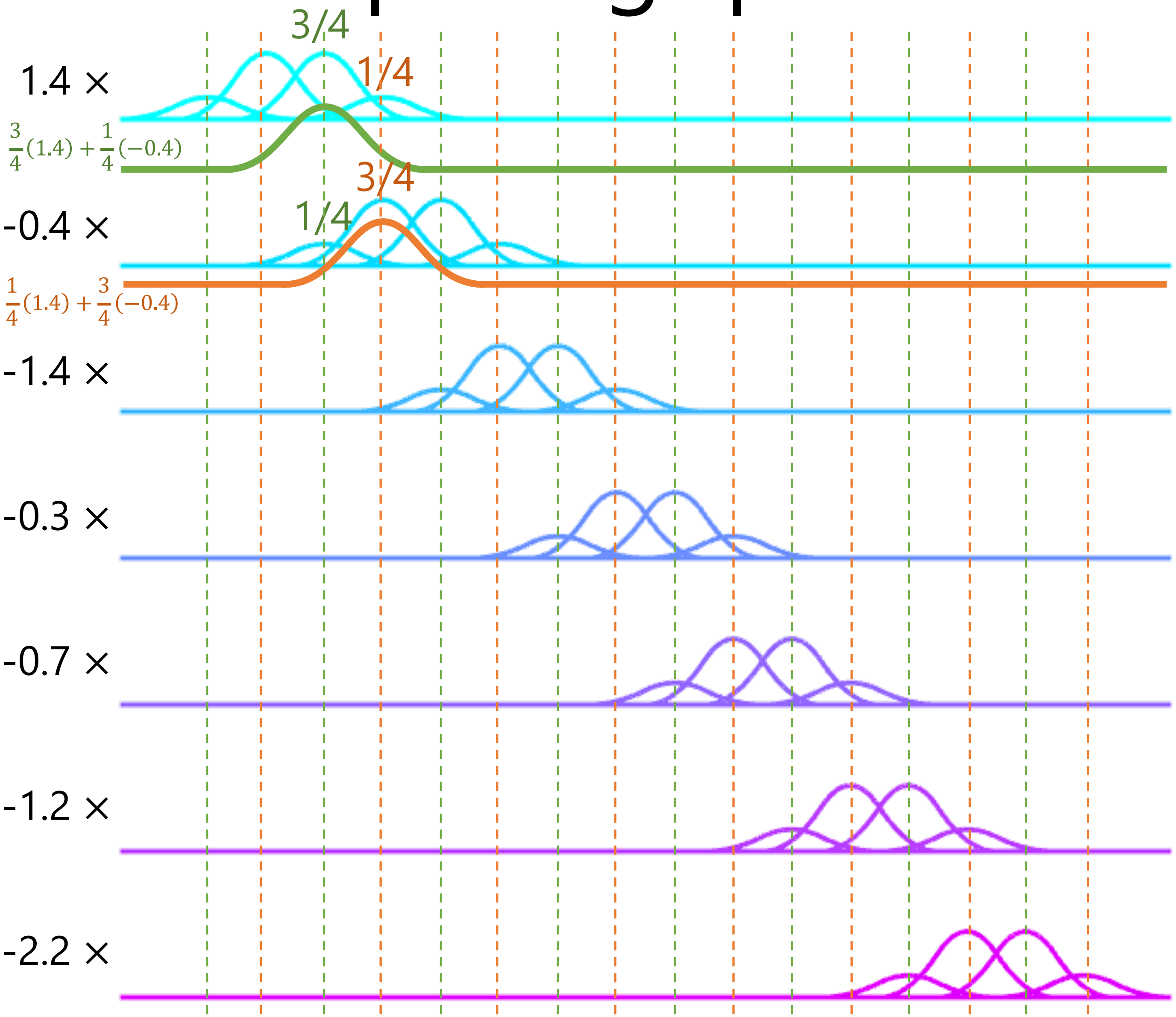
Quadratic

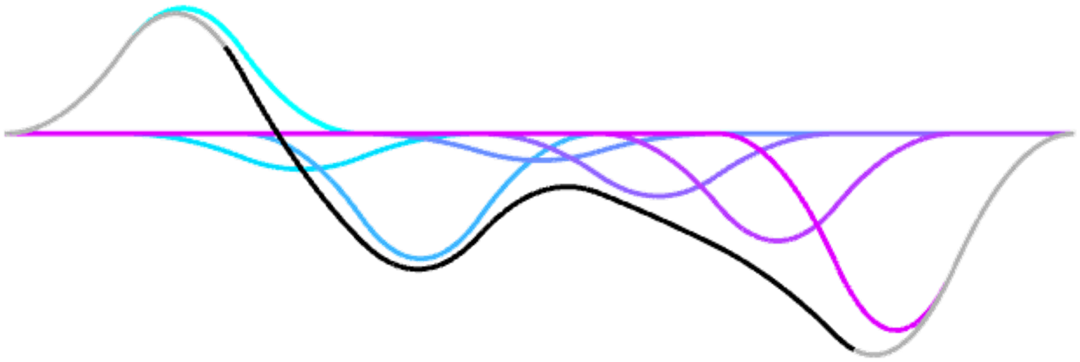


Decomposing quadratic B-spline



Decomposing quadratic B-spline





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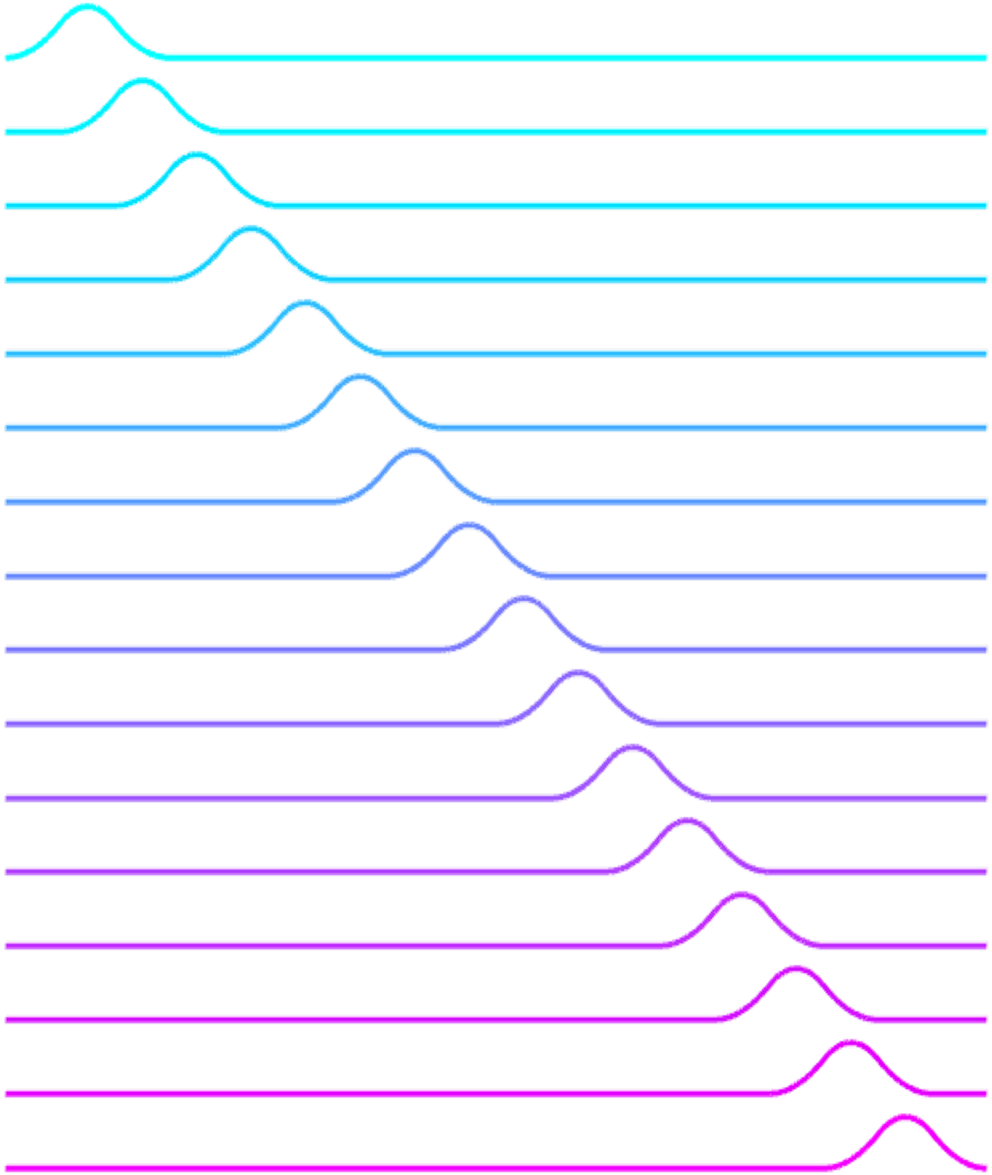

$\frac{1}{4}(1.4)$

$\frac{3}{4}(1.4)$

$\frac{3}{4}(1.4) + \frac{1}{4}(-0.4)$

$\frac{1}{4}(1.4) + \frac{3}{4}(-0.4)$

$\frac{3}{4}(-0.4) + \frac{1}{4}(-1.4)$

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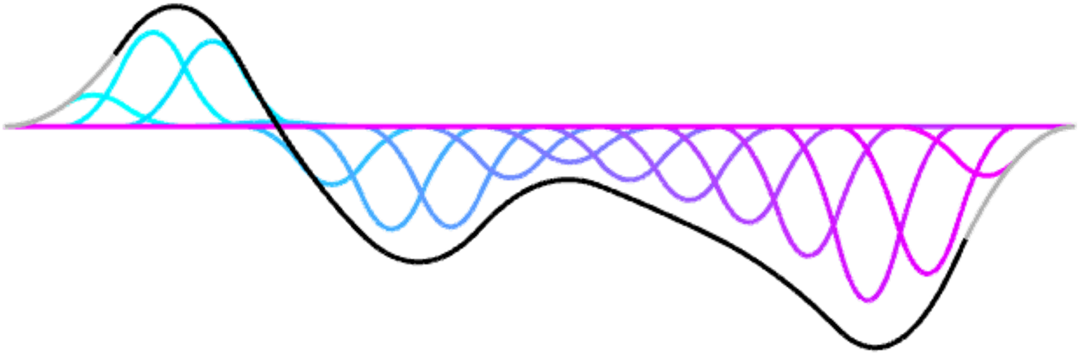
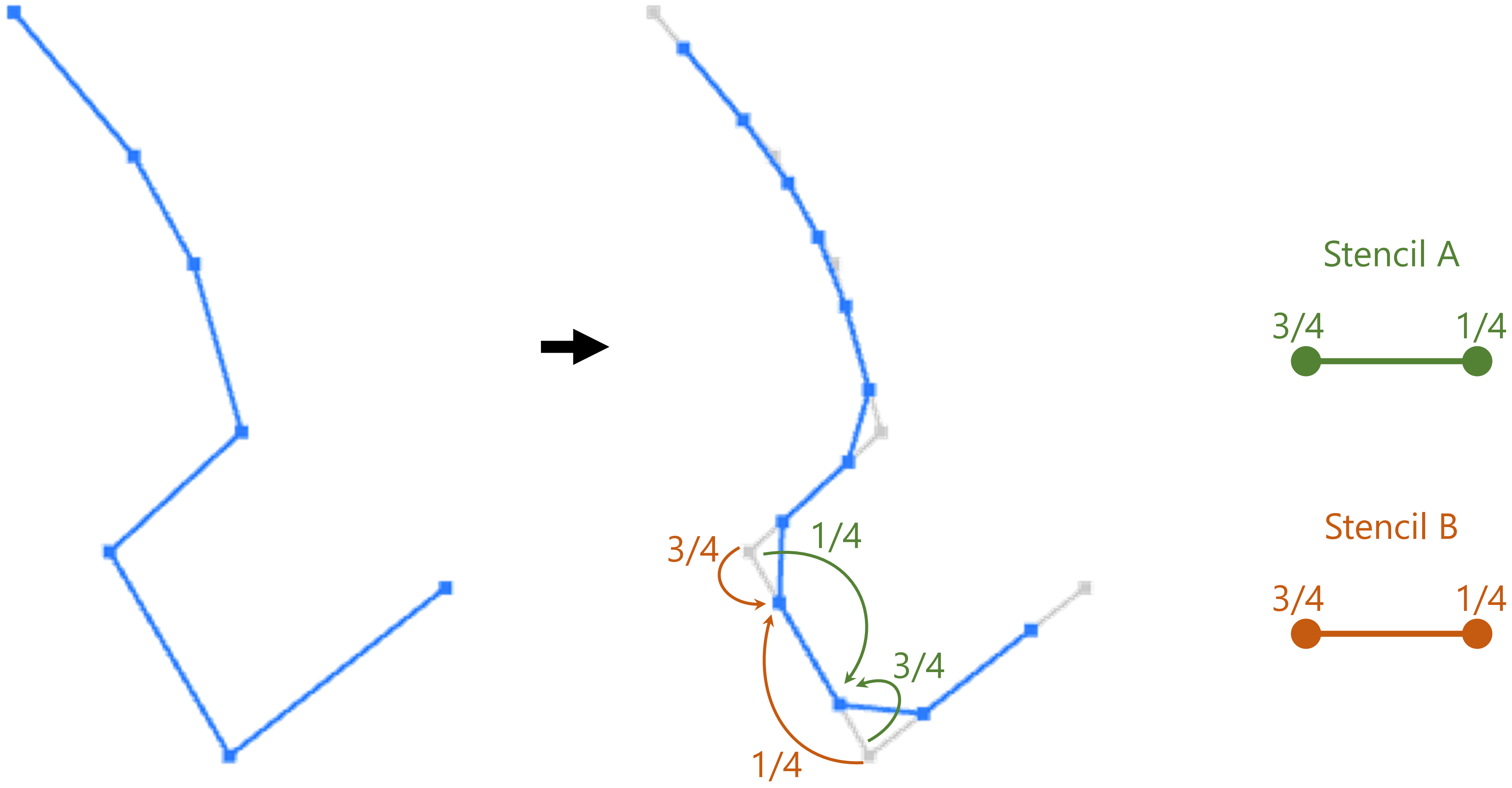
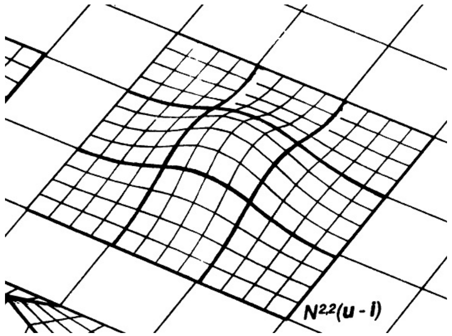


Generating quadratic curves via subdivision

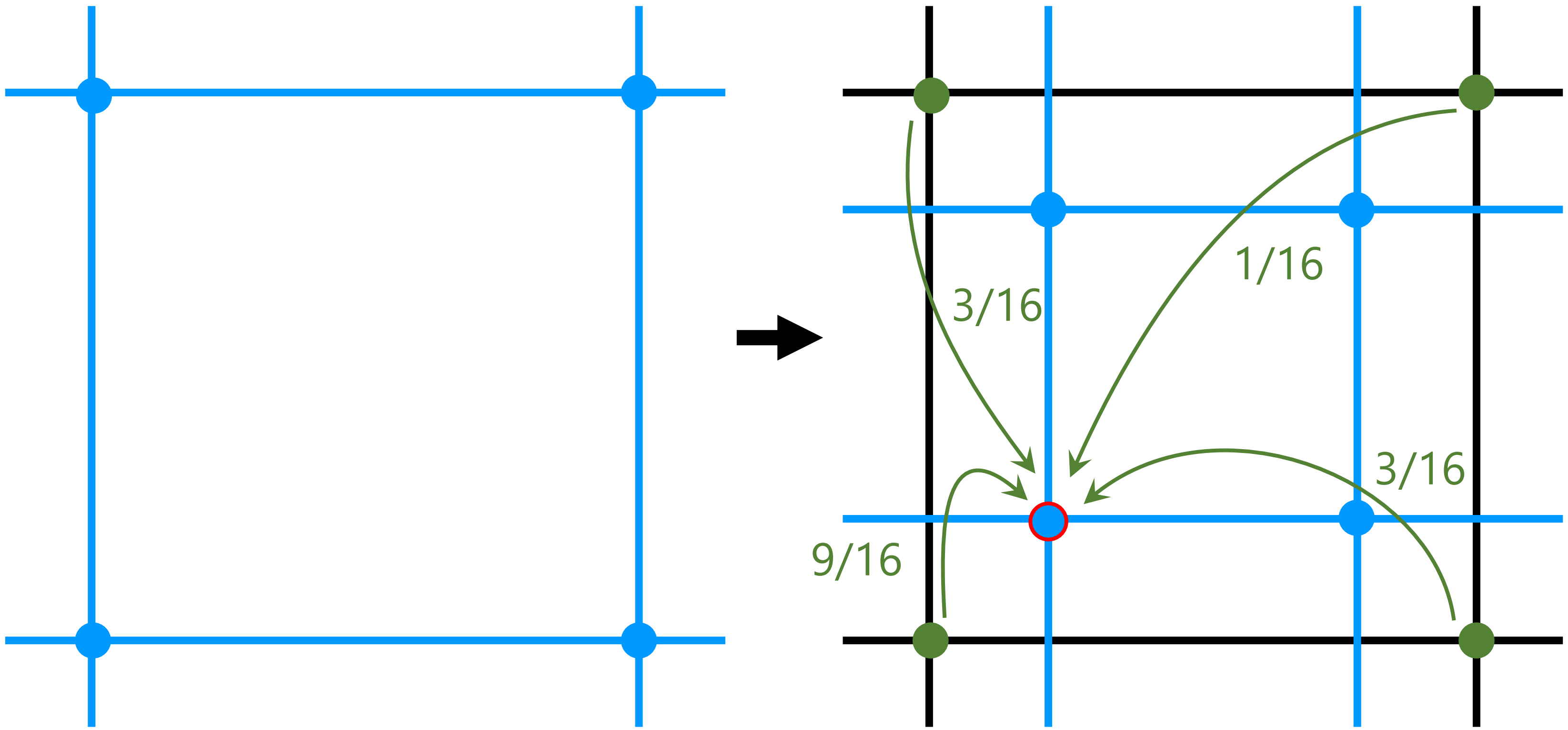


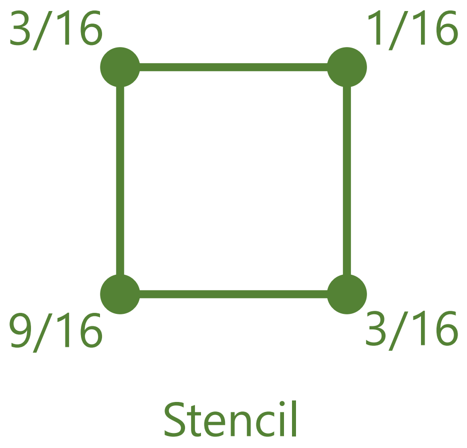
- Split each vertex into 2 new vertices
(= For each edge, generate 2 new vertices)

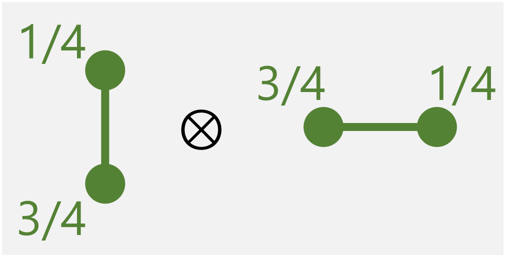
Generating quadratic surfaces via subdivision



Bi-quadratic basis:
$B_{2,2}(s,t) = B_2(s)\, B_2(t)$

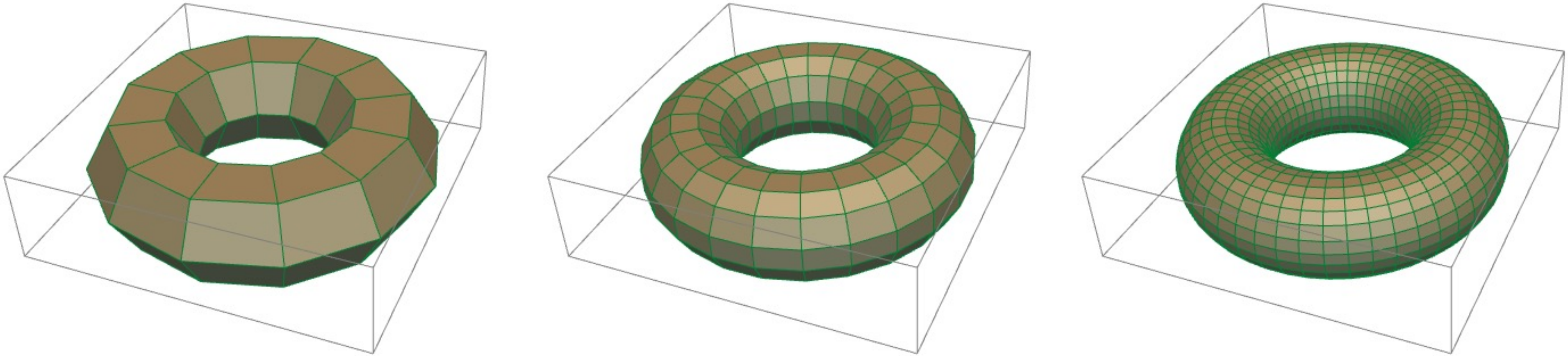




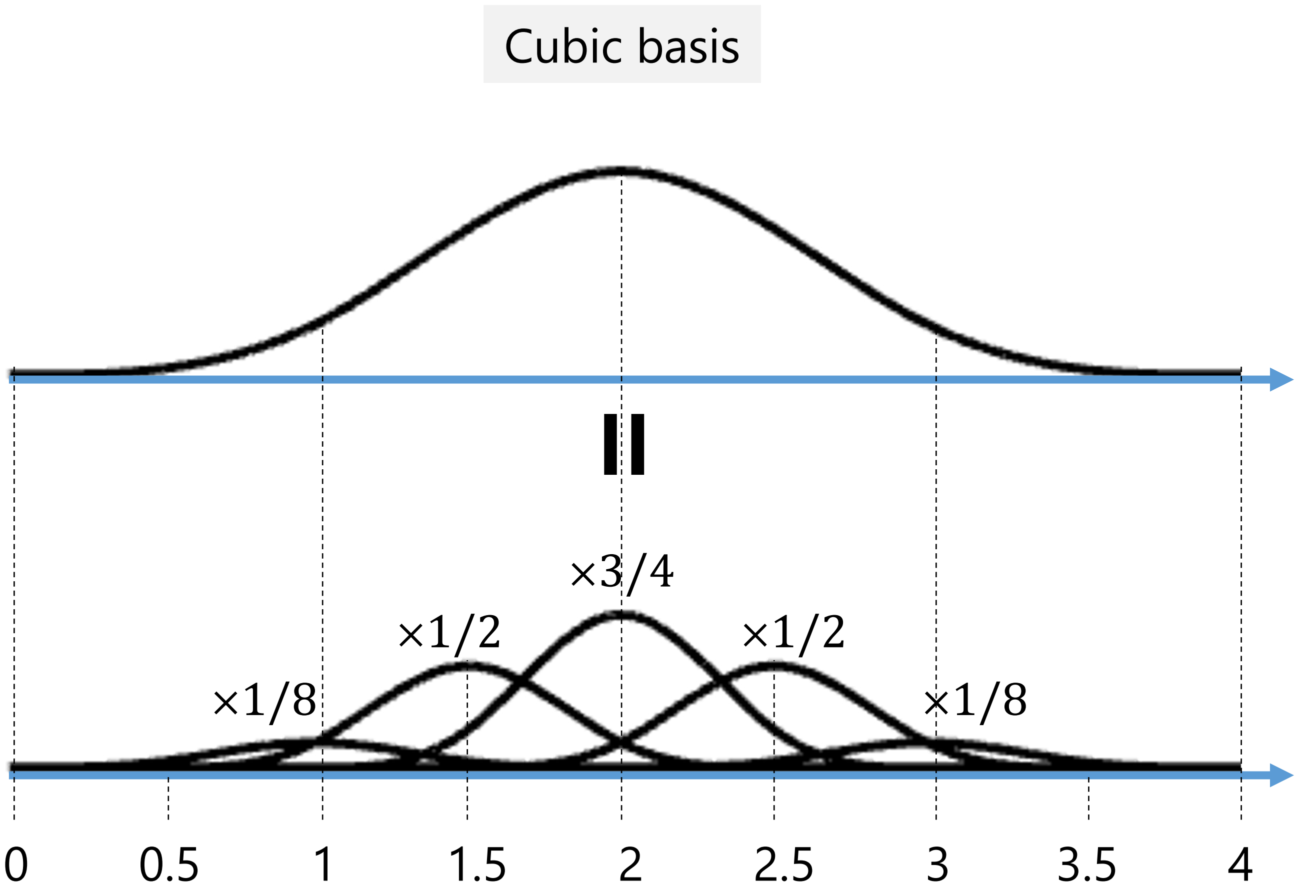


- Split each vertex into 4 new vertices
  (= For each face, generate 4 new vertices)

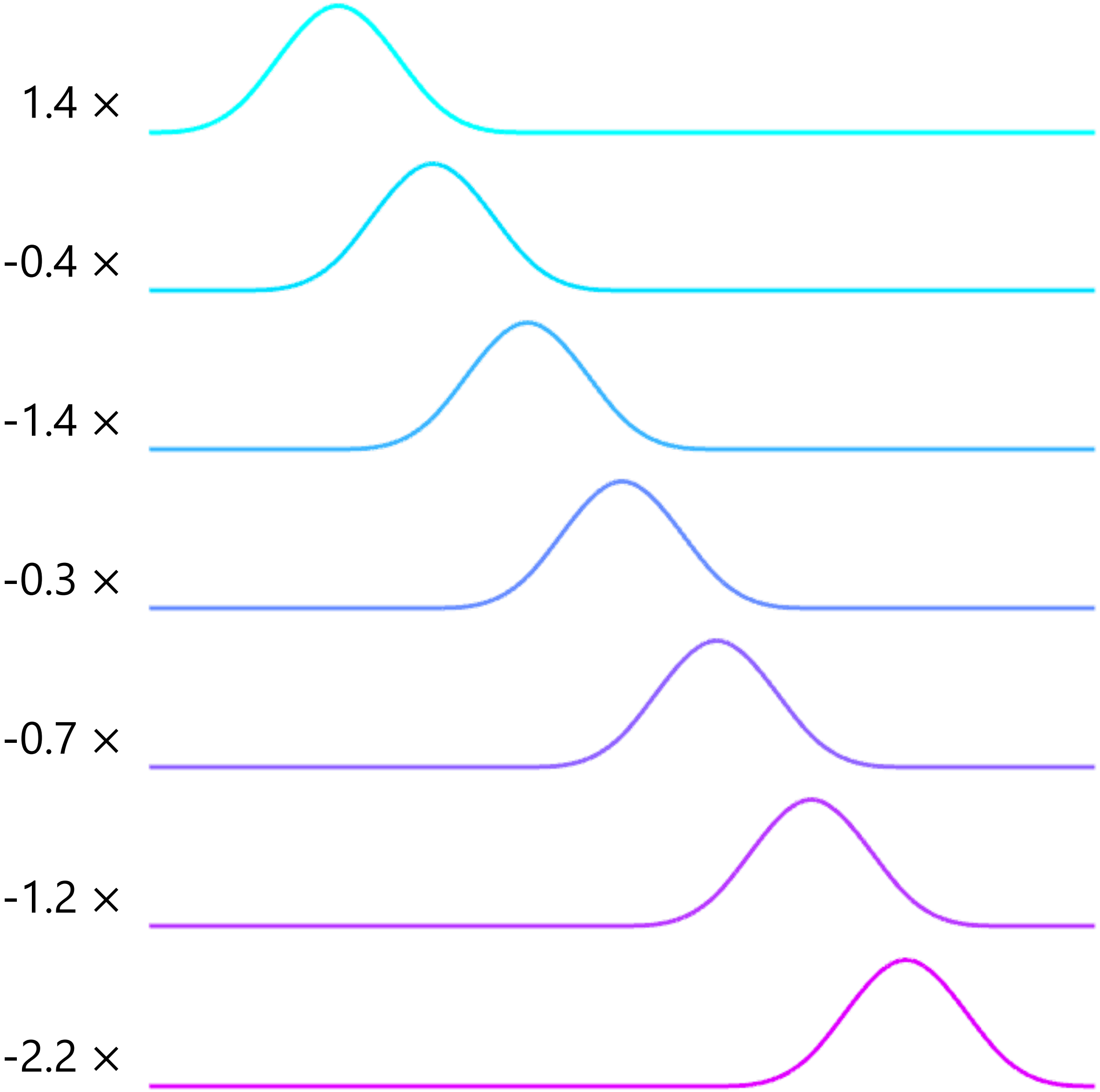
Subdividing a torus

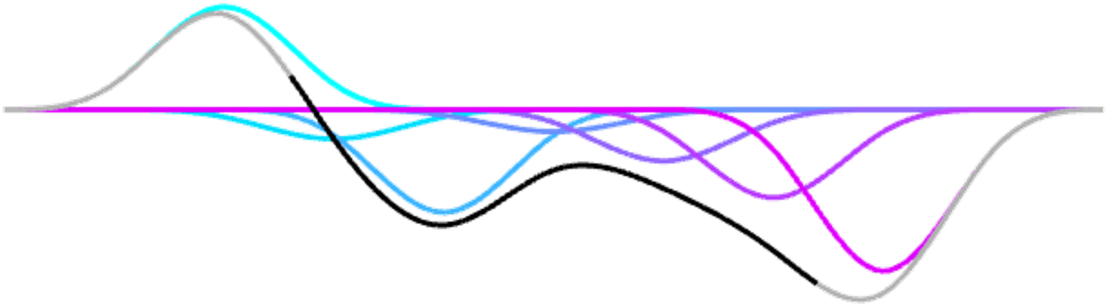


For the case of cubic B-spline



Decomposing cubic B-spline





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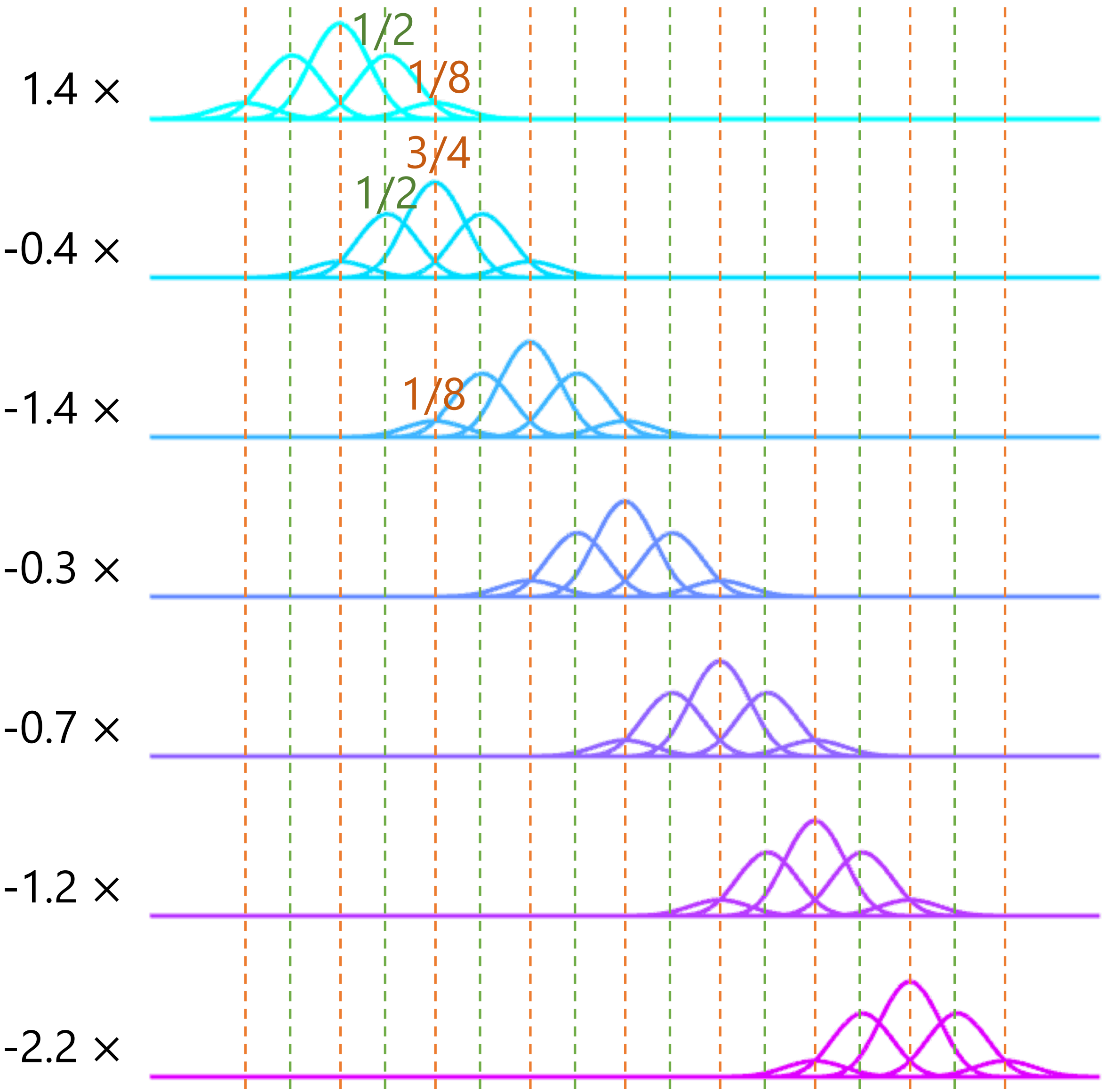

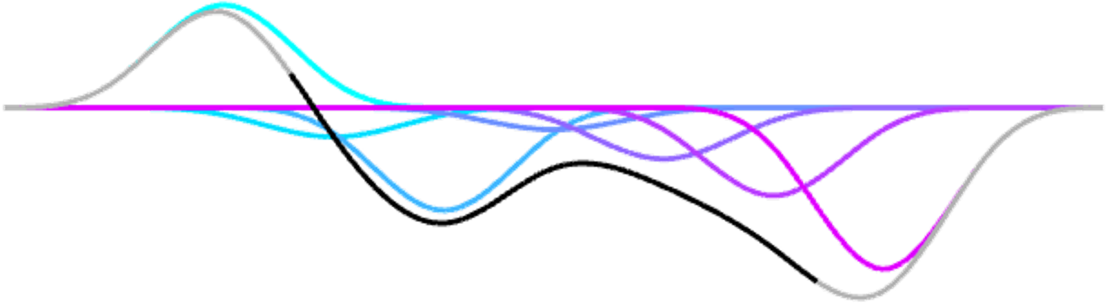

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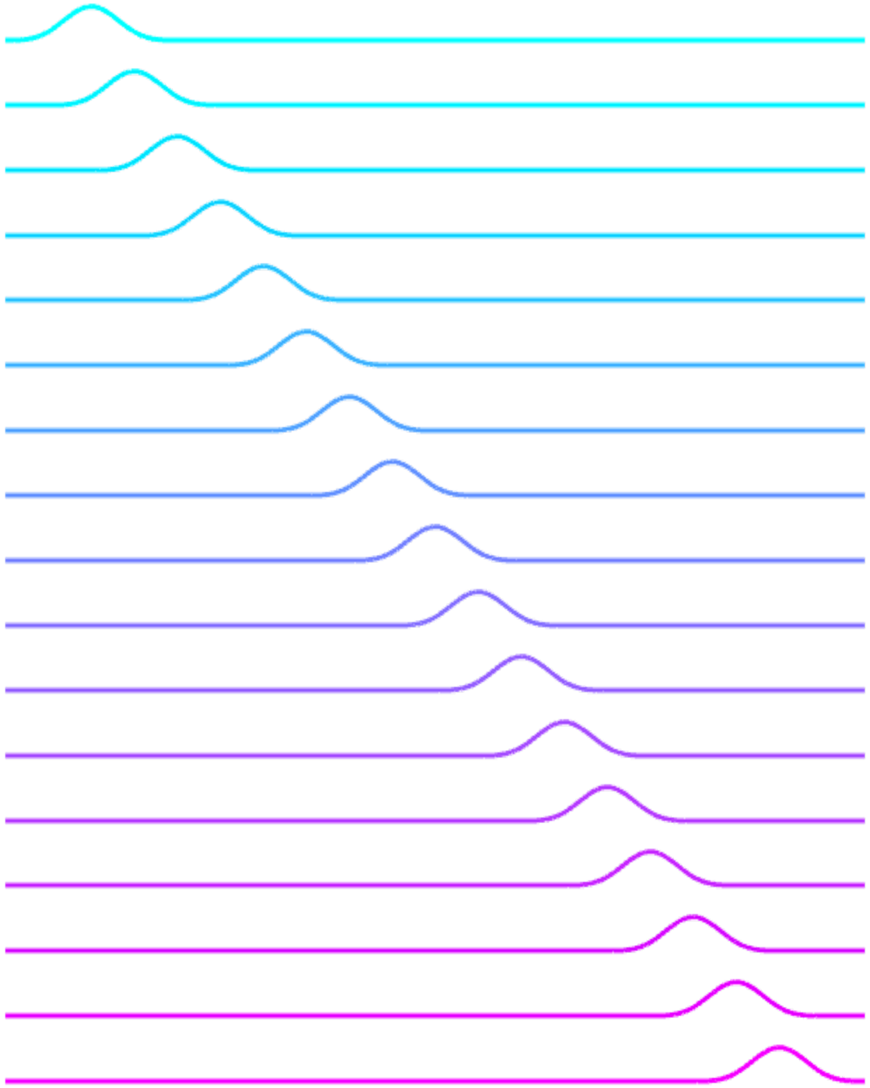

$\frac{1}{8}(1.4)$

$\frac{1}{2}(1.4)$

$\frac{3}{4}(1.4)+\frac{1}{8}(-0.4)$

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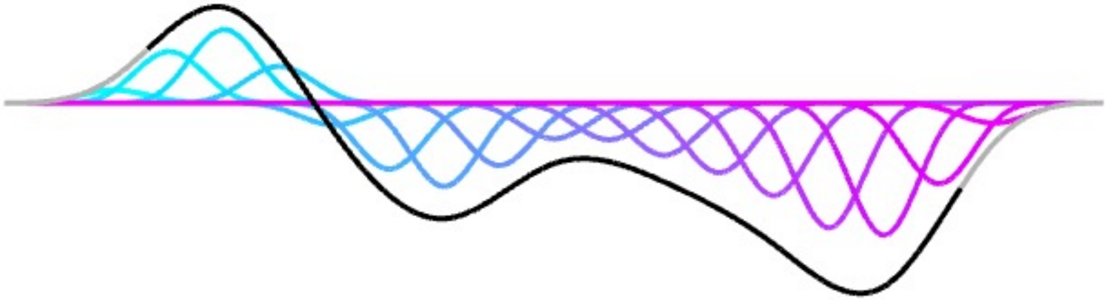
$\frac{1}{2}(-1.2)+\frac{1}{2}(-2.2)$

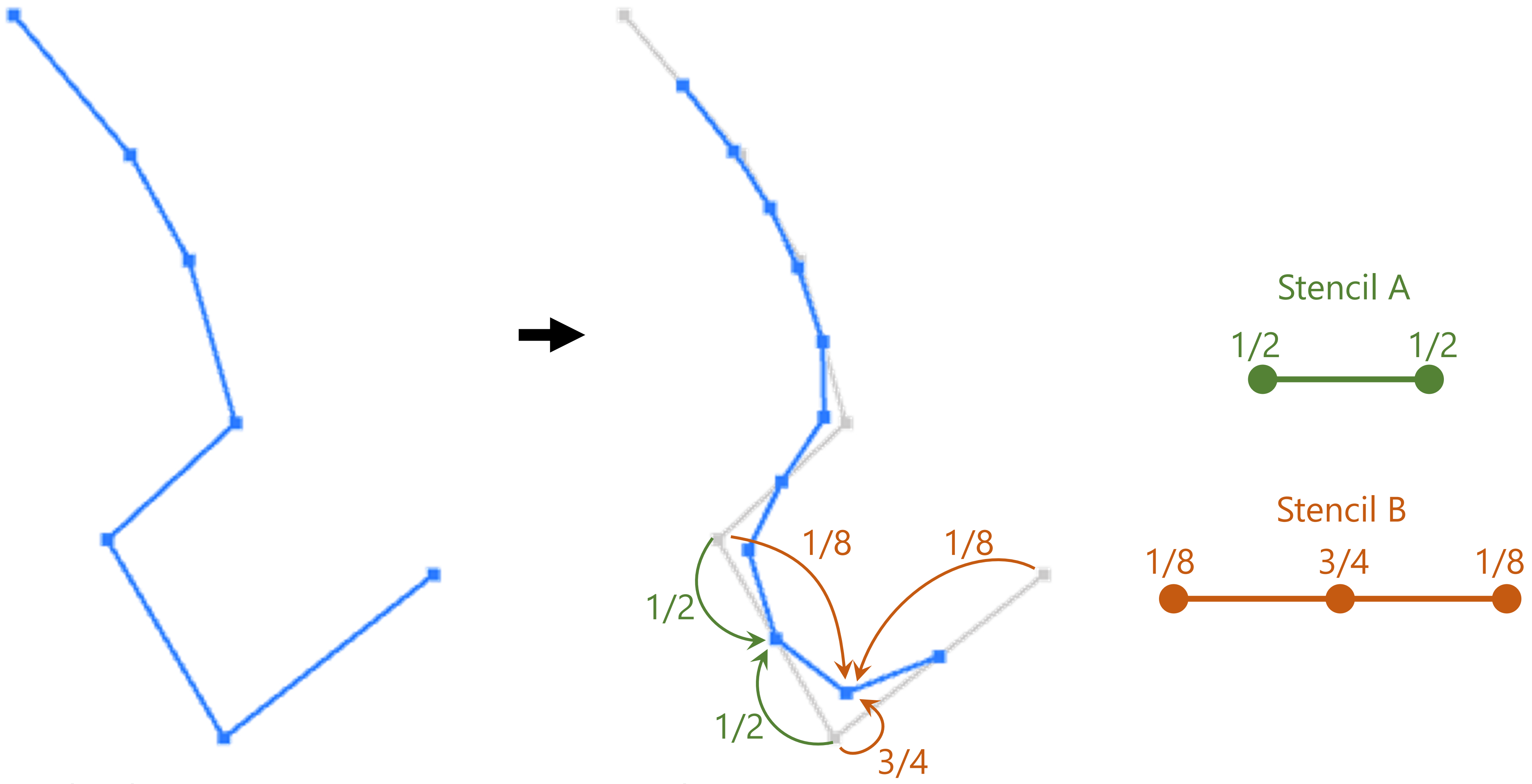
$\frac{1}{8}(-1.2)+\frac{3}{4}(-2.2)$

$\frac{1}{2}(-2.2)$

$\frac{1}{8}(-2.2)$

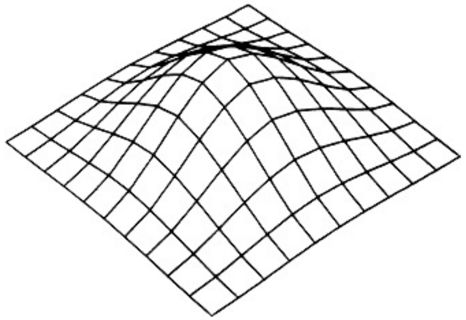


Generating cubic curves via subdivision

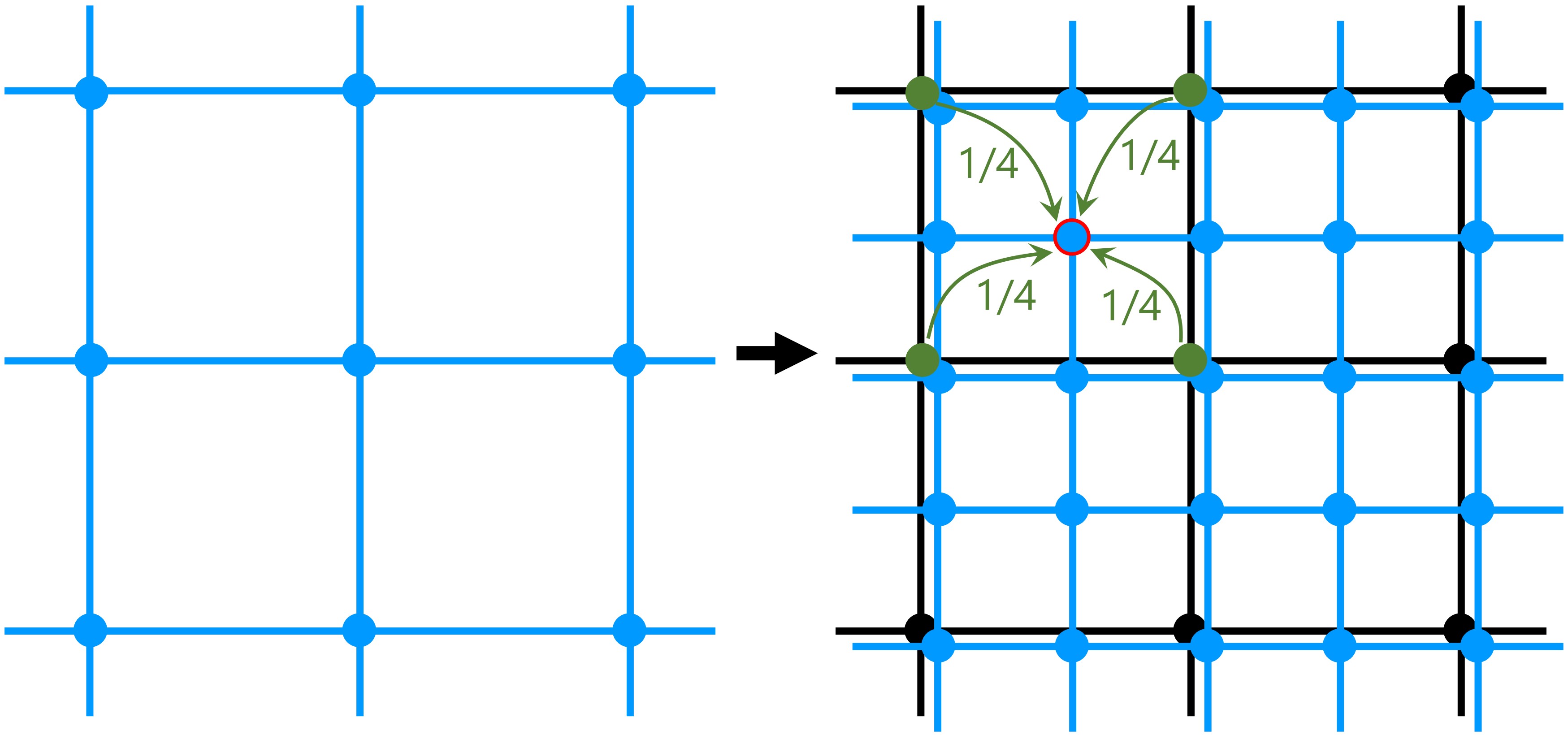


- For each edge, generate a new vertex at its midpoint
- Move each vertex to weighted average of its neighbors

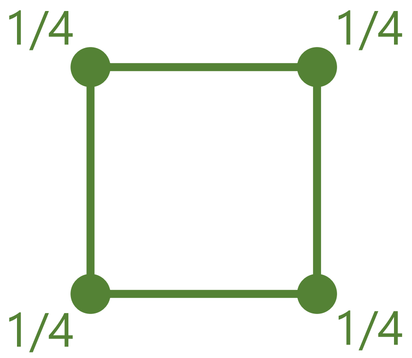
Generating cubic surfaces via subdivision

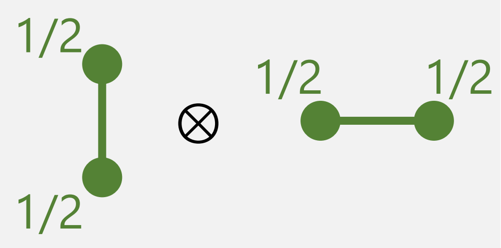


Bi-cubic basis:
$B_{3,3}(s,t) = B_3(s)\, B_3(t)$



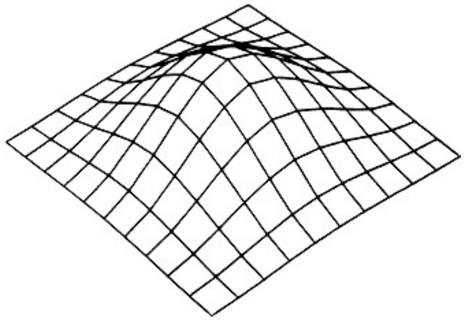
“face point” stencil





- For each face, generate a new vertex at its barycenter

Generating cubic surfaces via subdivision



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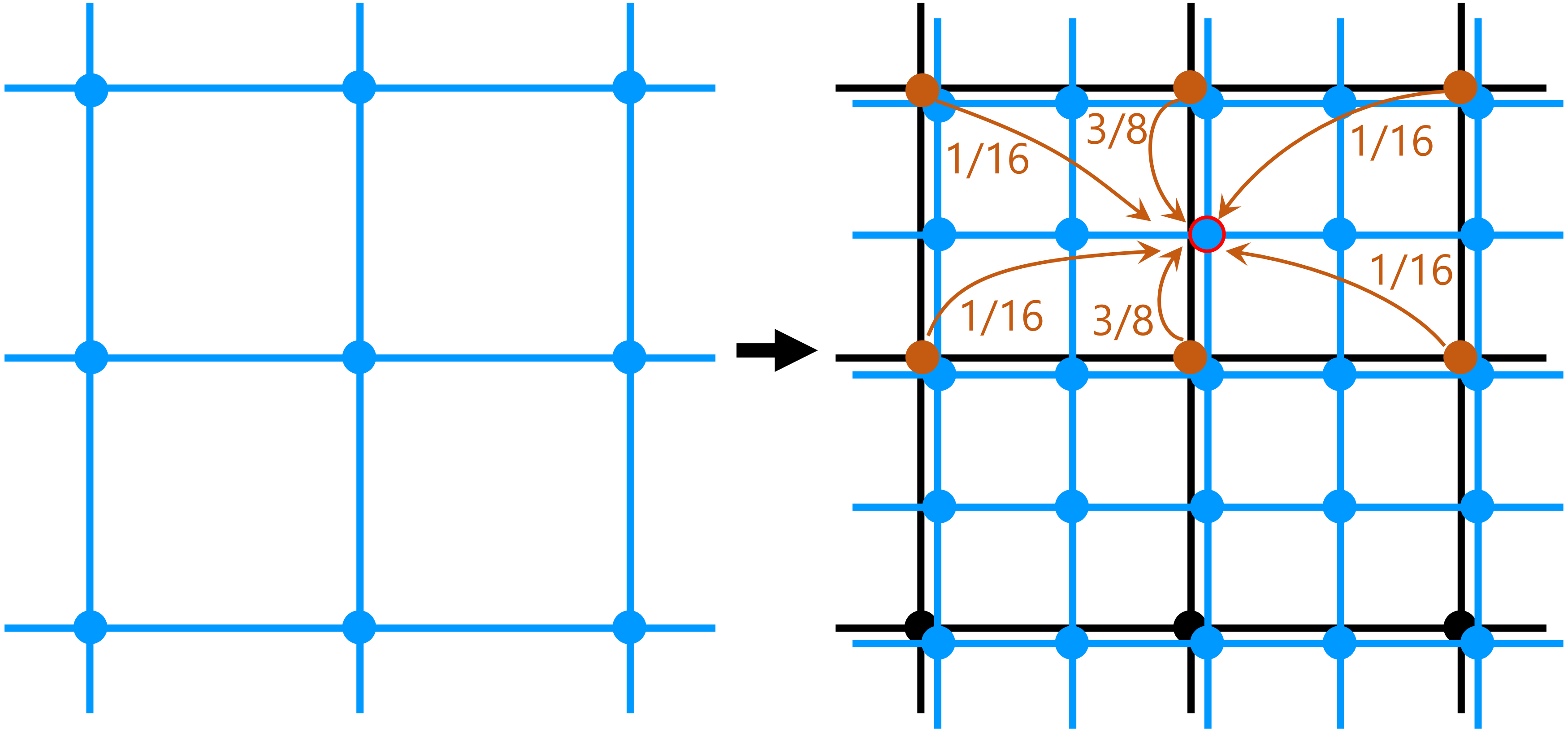
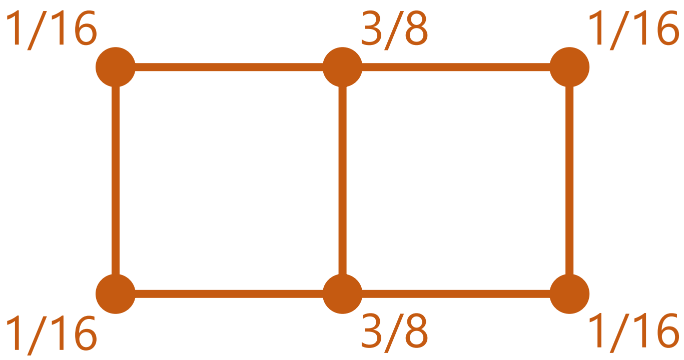
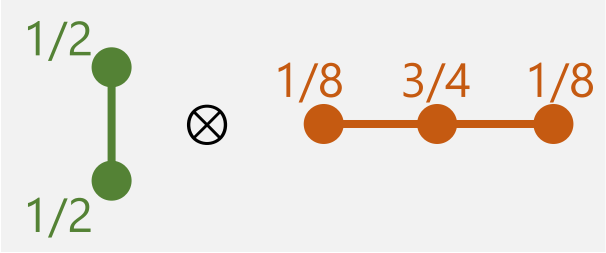


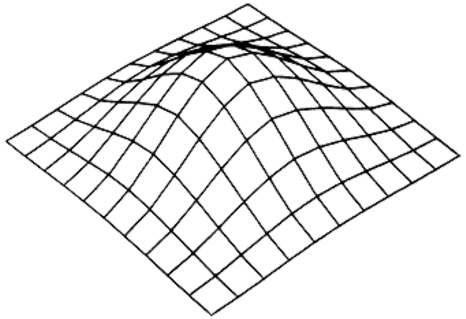
“edge point” stencil





- For each edge, generate a new vertex at weighted average of its neighbors

Generating cubic surfaces via subdivision



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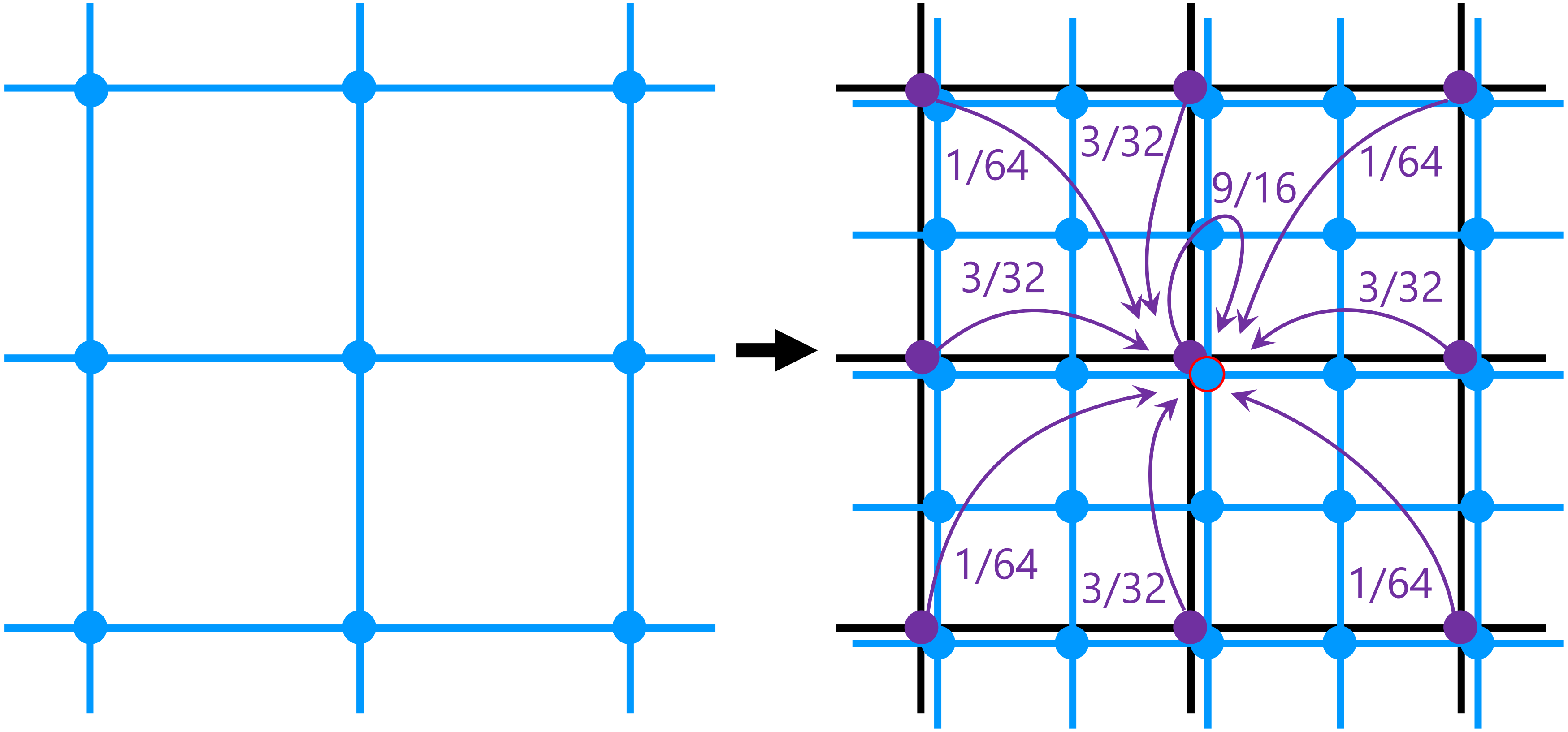
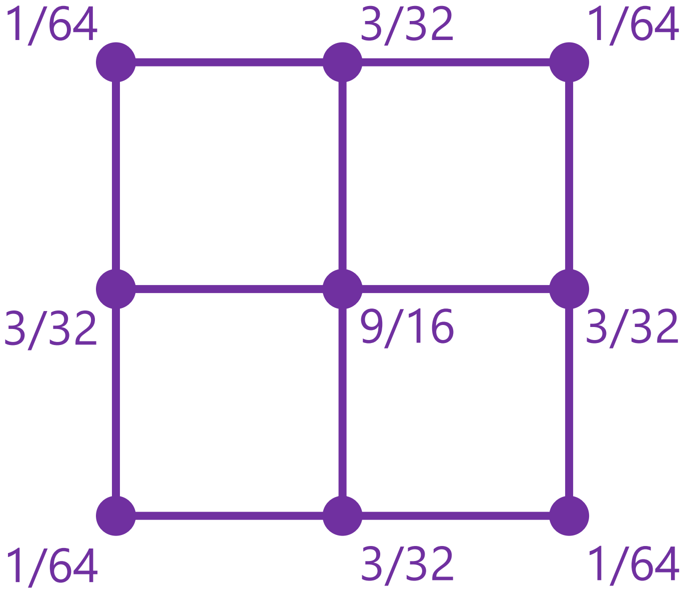
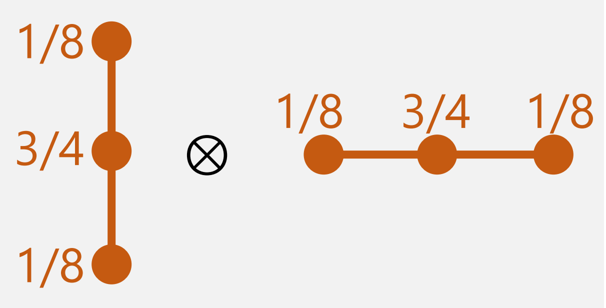


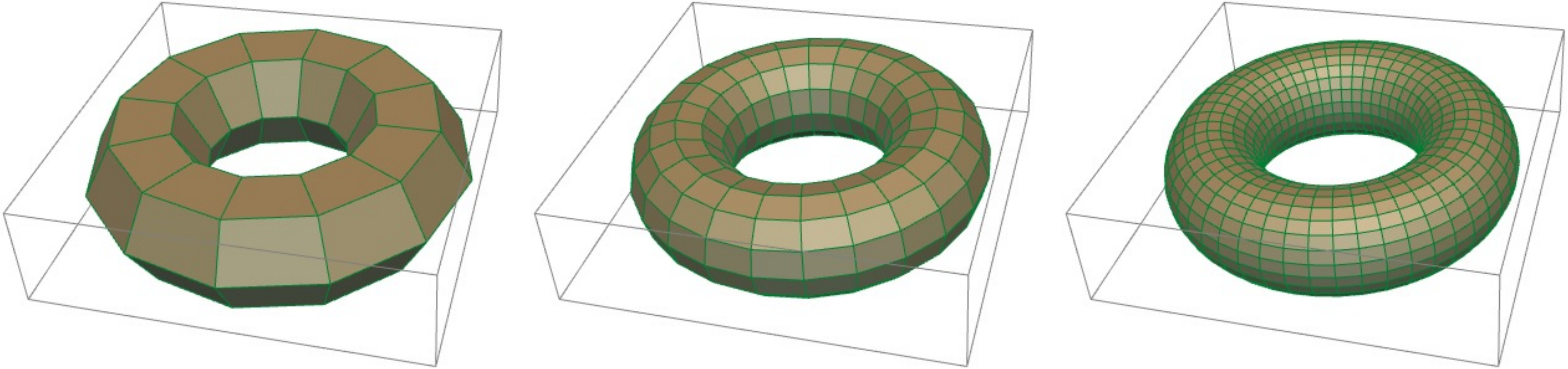
“vertex point” stencil





- Move each vertex to weighted average of its neighbors

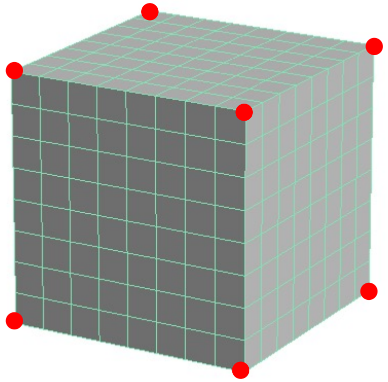
Subdividing a torus

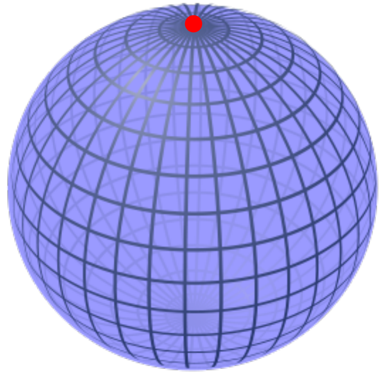


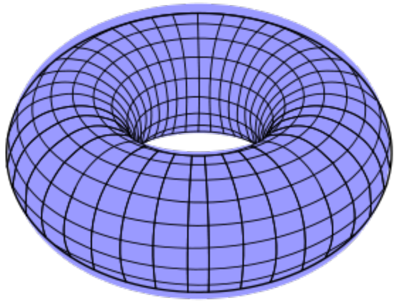
Generalizing subdivision scheme

Assumption in the aforementioned formulation

- “Clean” quadrilateral decomposition of the region
  - “Clean” vertex: # of neighboring faces (valence) is 4
    - If valence is not 4 ➔ singularity

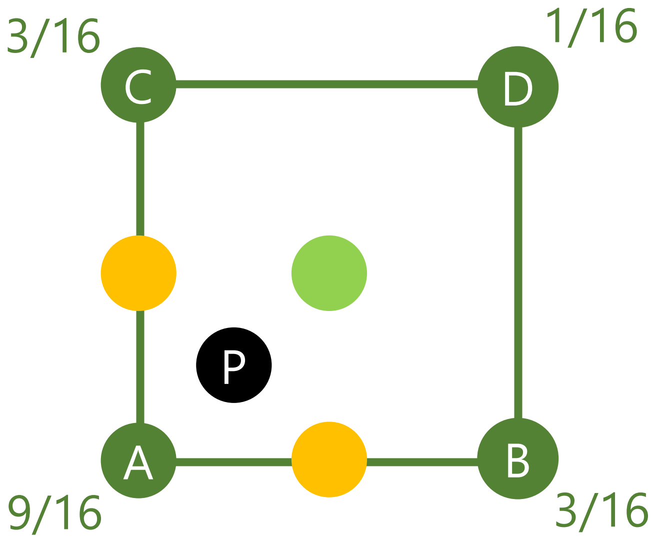






- Generally impossible to obtain except special cases (torus)
- Strength of subdivision schemes: applicable to singularities
  - Generalize stencils through geometric interpretations

Generalizing quadratic stencil (Doo-Sabin)



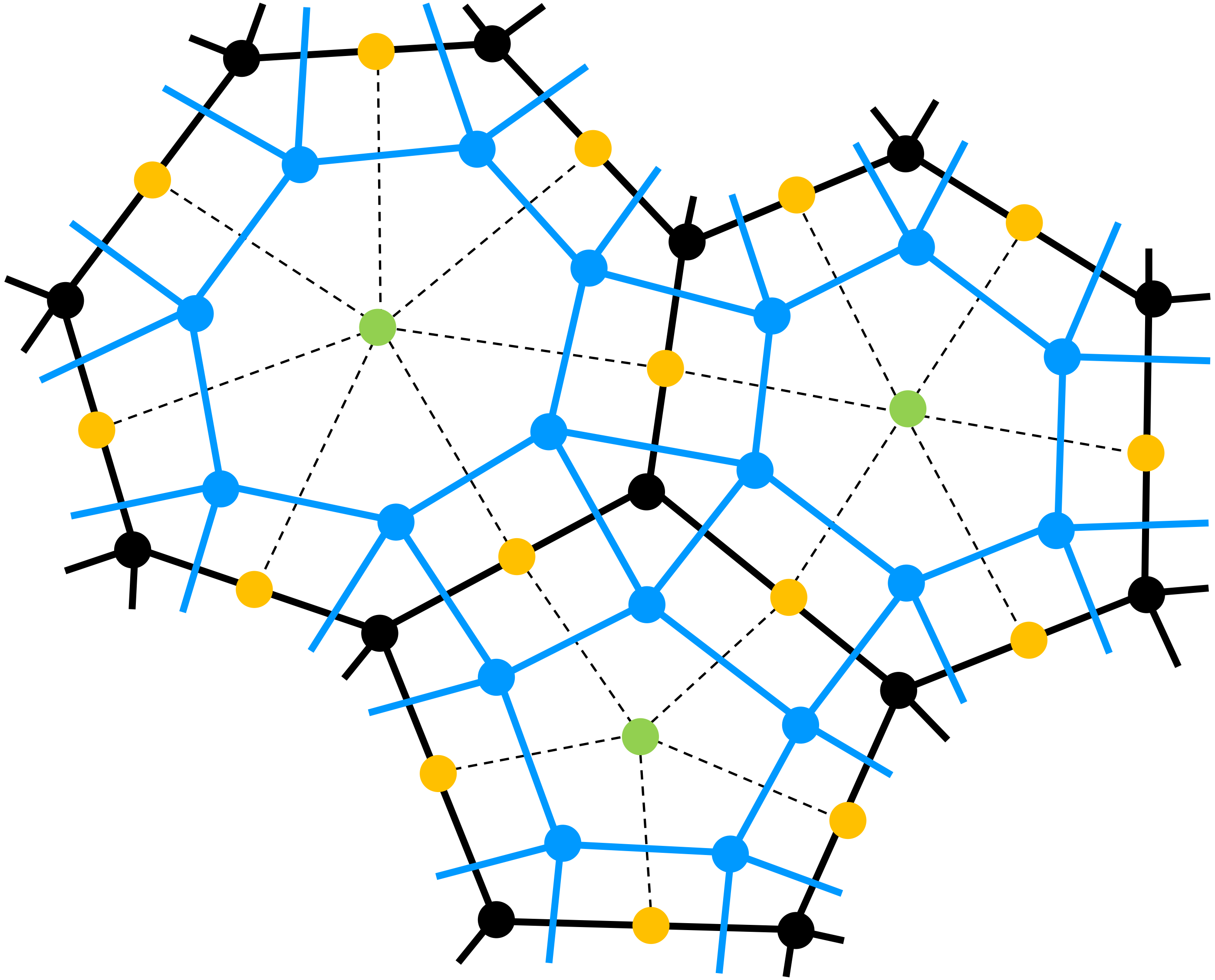
$$P = \frac{1}{16}(9\,A + 3\,B + 3\,C + D)$$

$$= \frac{\overset{\text{Barycenter}}{\frac{A+B+C+D}{4}} + \overset{\text{Midpoint}}{\frac{A+B}{2}} + \overset{\text{Midpoint}}{\frac{A+C}{2}} + A}{4}$$

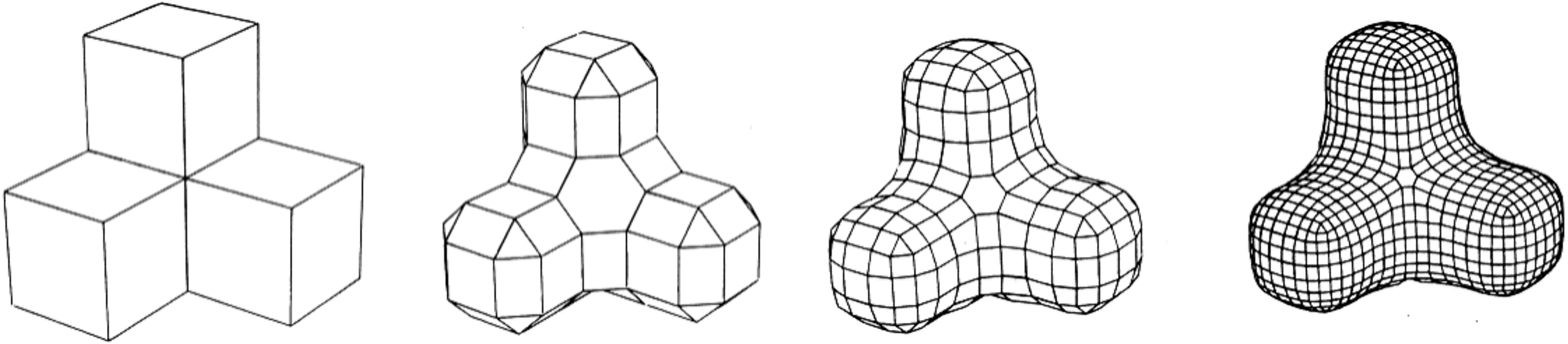
For each polygon’s each vertex, generate a new vertex at the average of the polygon’s barycenter, its adjacent edges’ midpoints, and itself

➔ Applicable to general polygon mesh

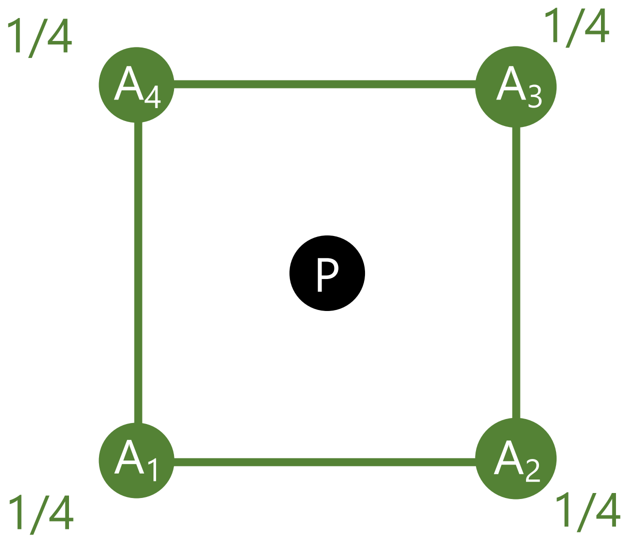
Examples of Doo-Sabin



Examples of Doo-Sabin



Generalizing cubic stencils (Catmull-Clark)

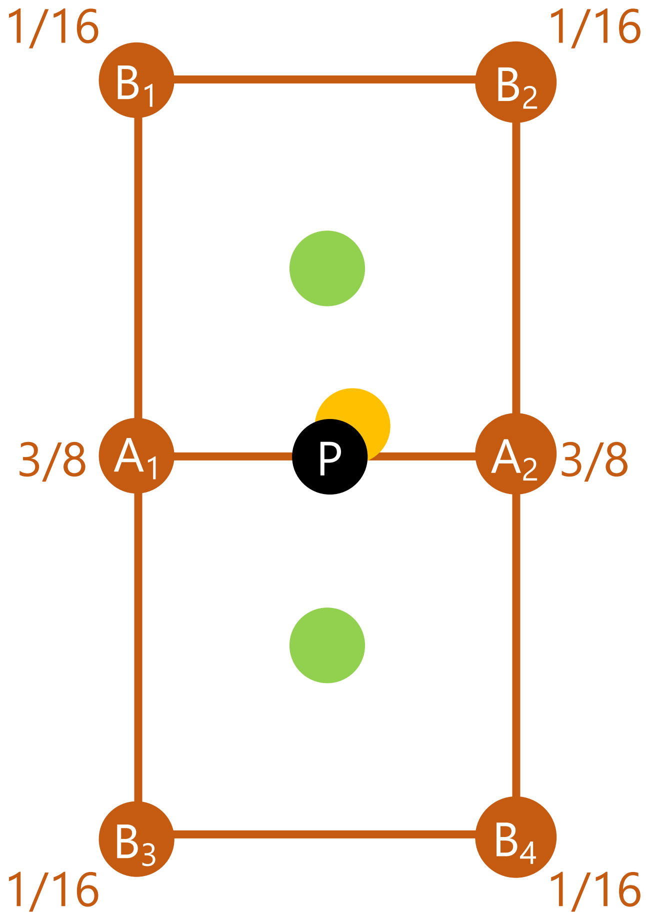


$$P = \frac{A_1 + A_2 + A_3 + A_4}{4}$$

For each polygon, generate a new vertex at its barycenter

➔ Applicable to general polygon mesh

Generalizing cubic stencils (Catmull-Clark)



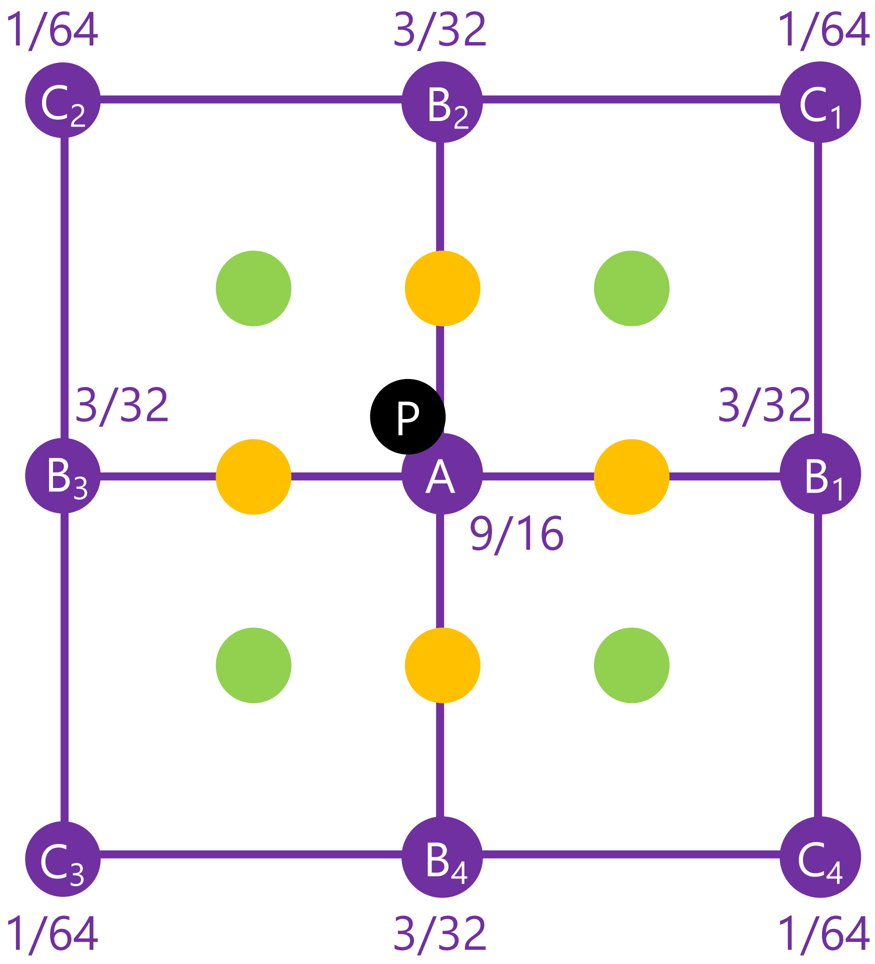
$$P = \frac{3}{8}(A_1 + A_2) + \frac{1}{16}(B_1 + B_2 + B_3 + B_4)$$

$$= \frac{\frac{\overset{\text{Barycenter}}{\frac{A_1 + A_2 + B_1 + B_2}{4}} + \overset{\text{Barycenter}}{\frac{A_1 + A_2 + B_3 + B_4}{4}}}{2} + \overset{\text{Midpoint}}{\frac{A_1 + A_2}{2}}}{2}$$

For each edge, generate a new vertex at the average of the barycenters of its adjacent polygons and its midpoint

➔ Applicable to general polygon mesh

Generalizing cubic stencils (Catmull-Clark)



$$P = \frac{9}{16}A + \frac{3}{32}(B_1 + B_2 + B_3 + B_4) + \frac{1}{64}(C_1 + C_2 + C_3 + C_4)$$

$$= \frac{1}{4}\left\{\underbrace{\frac{\overset{\text{Barycenter}}{\frac{A + B_1 + C_1 + B_2}{4}} + \overset{\text{Barycenter}}{\frac{A + B_2 + C_2 + B_3}{4}} + \overset{\text{Barycenter}}{\frac{A + B_3 + C_3 + B_4}{4}} + \overset{\text{Barycenter}}{\frac{A + B_4 + C_4 + B_1}{4}}}{4}}_{Q}\right\}$$

$$+ \frac{2}{4}\left\{\underbrace{\frac{\overset{\text{Midpoint}}{\frac{A + B_1}{2}} + \overset{\text{Midpoint}}{\frac{A + B_2}{2}} + \overset{\text{Midpoint}}{\frac{A + B_3}{2}} + \overset{\text{Midpoint}}{\frac{A + B_4}{2}}}{4}}_{R}\right\}$$

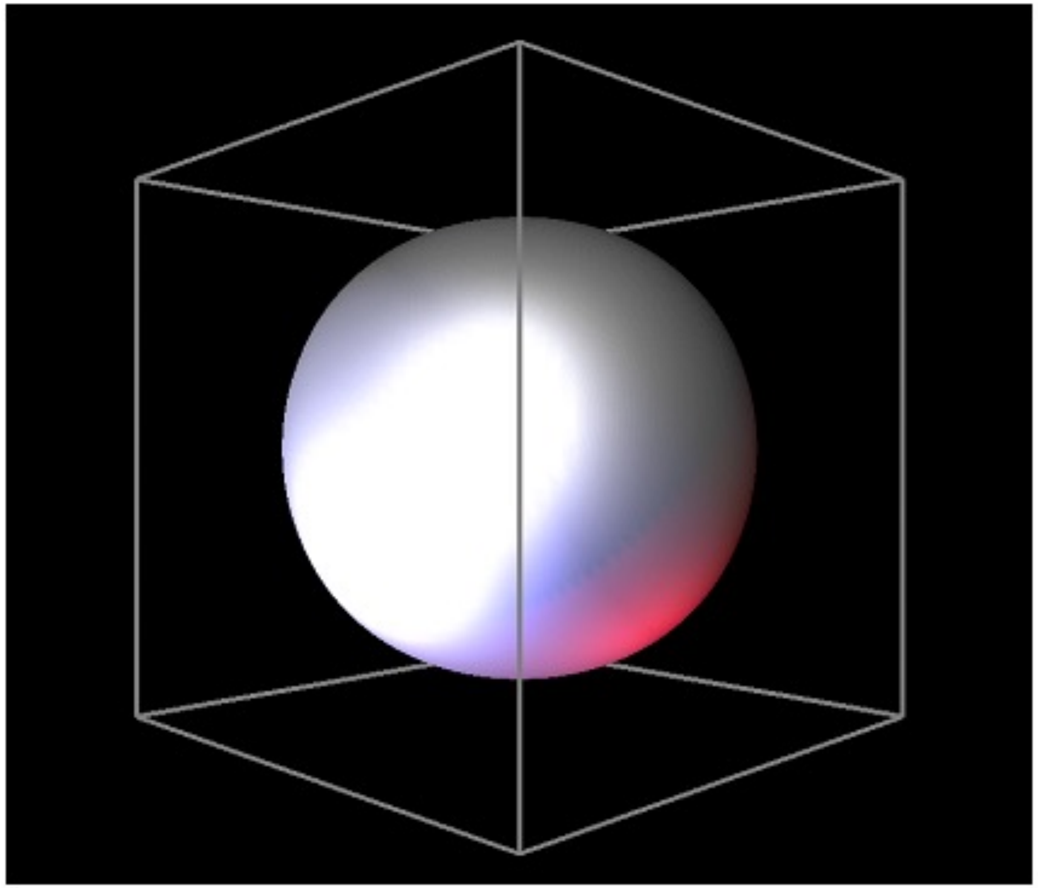
$$+ \frac{1}{4}A$$

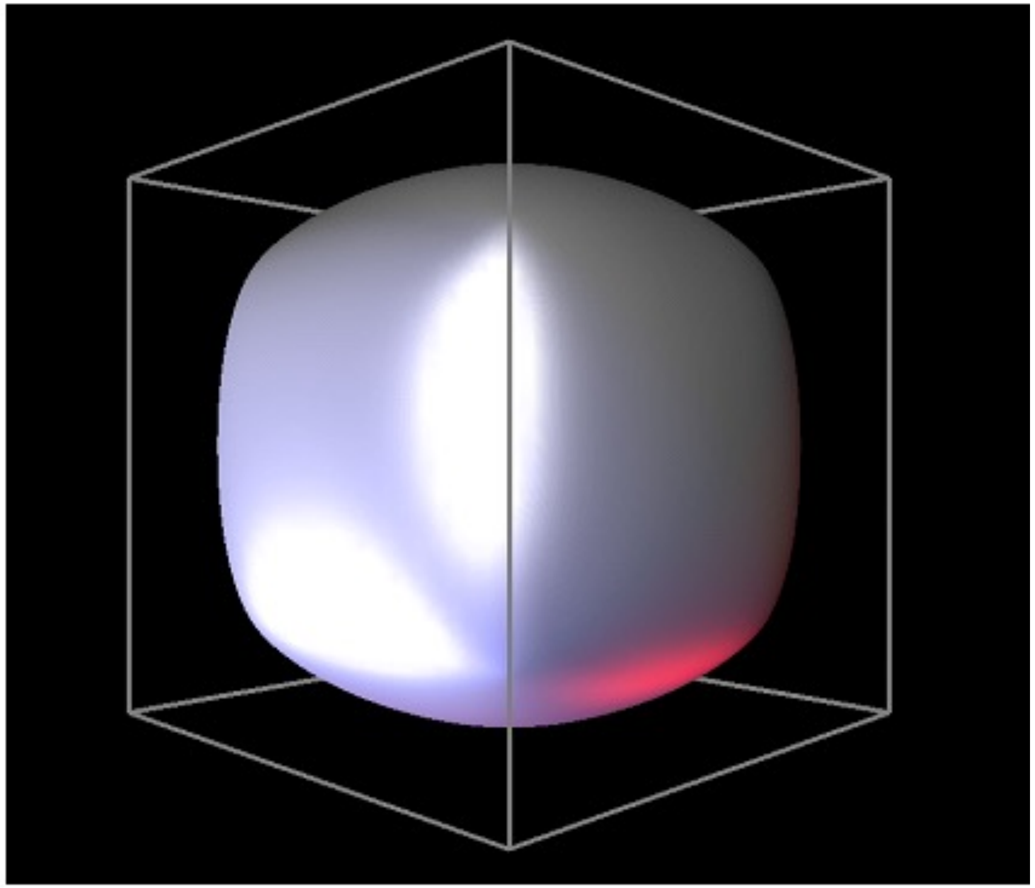
When A's valence is n,

$$P = \frac{1}{n}Q + \frac{2}{n}R + \frac{n-3}{n}A$$

➔ Applicable to general polygon mesh

Comparison



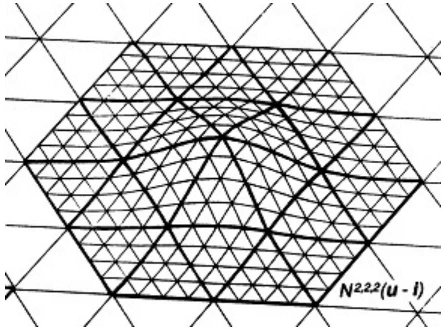


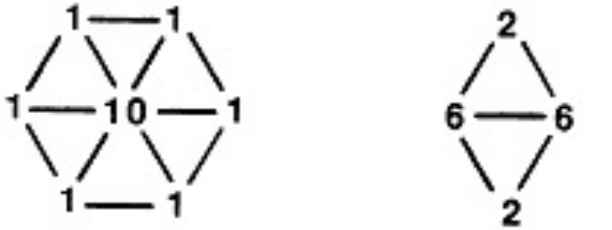
Catmull-Clark = cubic surface

Doo-Sabin = quadratic surface

Subdivision scheme for triangle meshes (Loop)

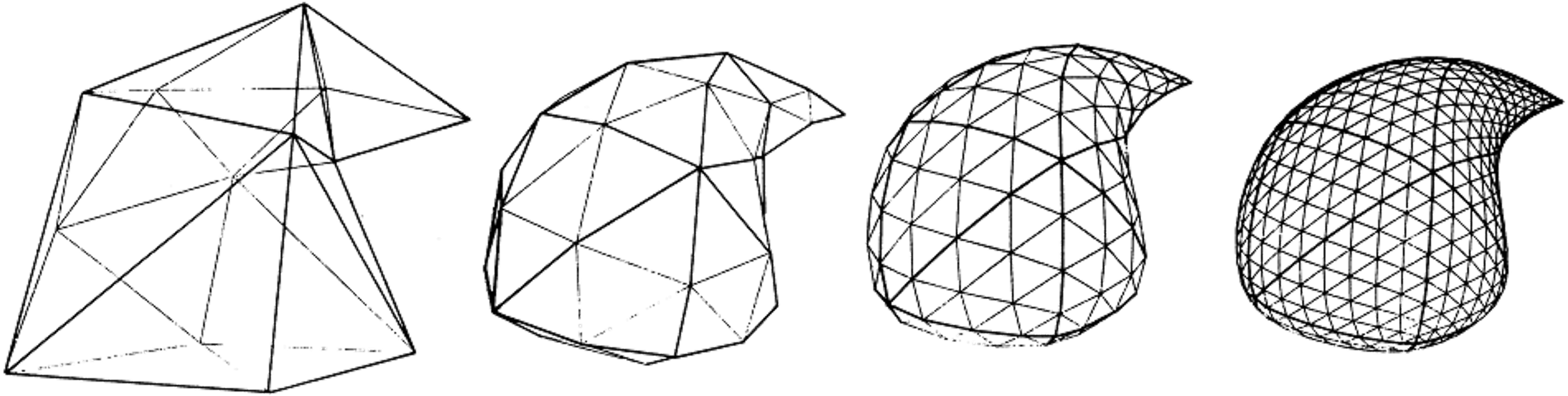
- Based on B-spline basis defined on triangular lattice



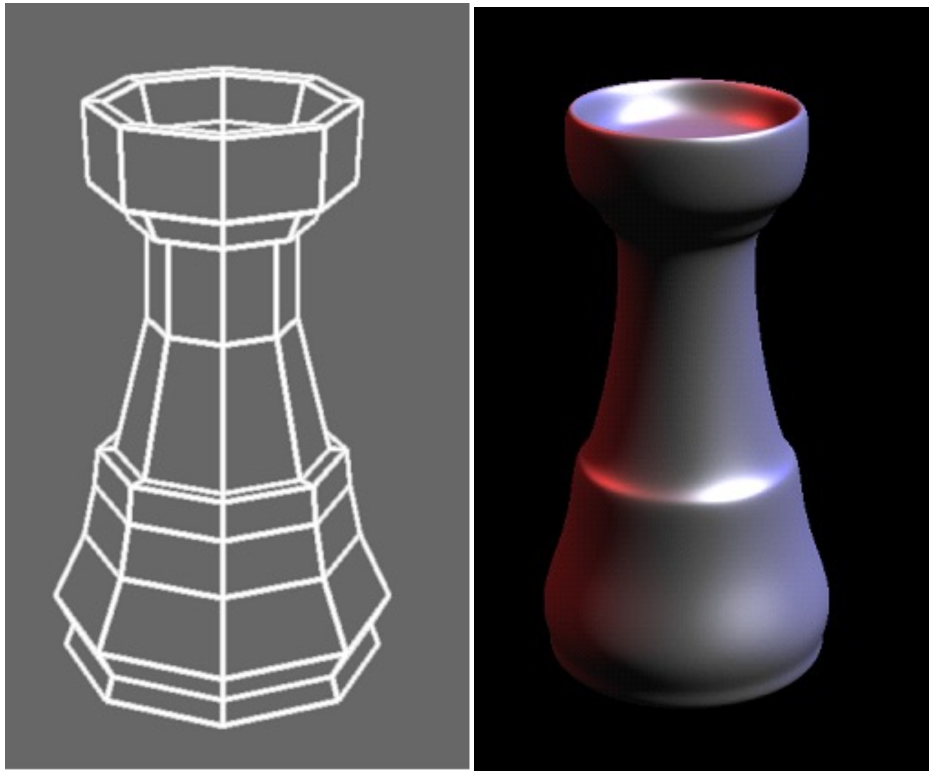


When valence isn't 6 ➔ derived from complicated analysis (see Loop's paper)

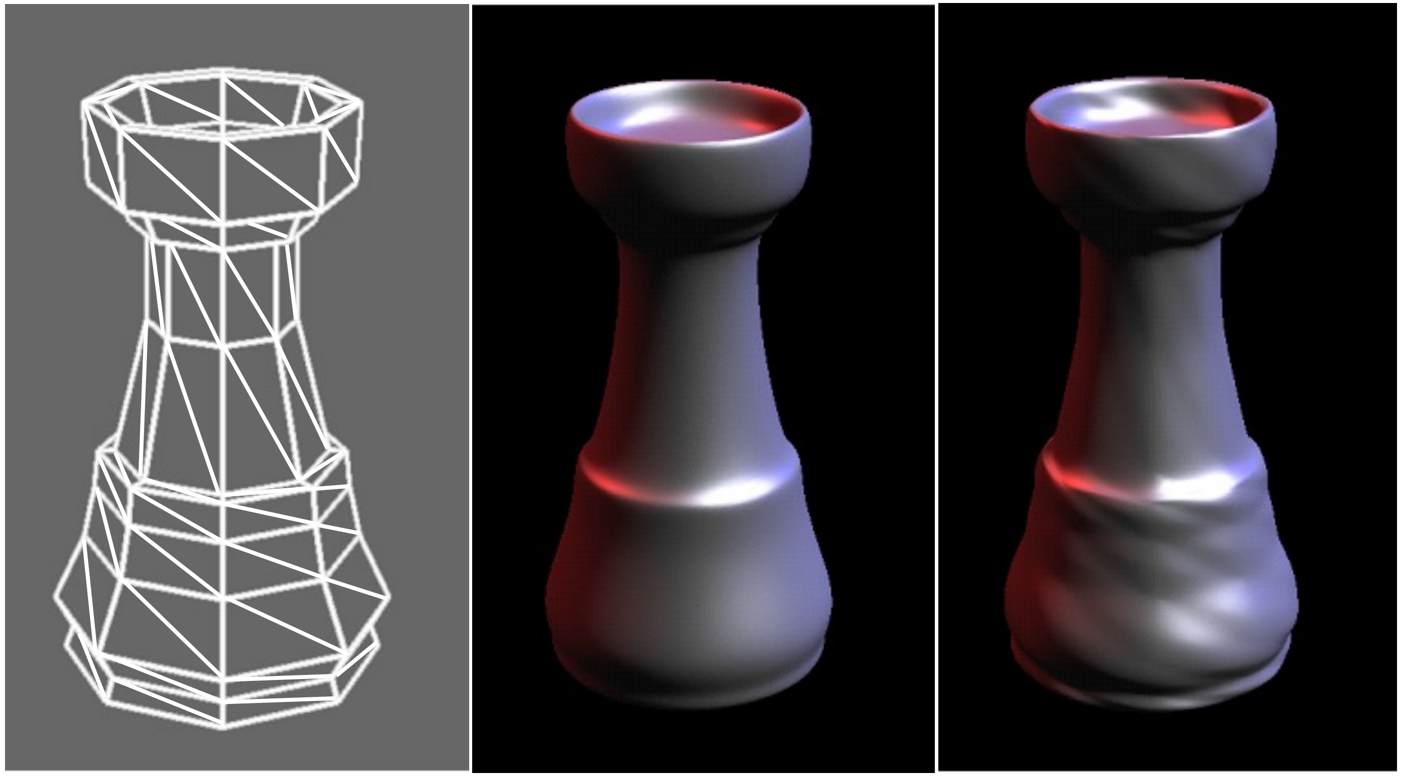
- C^2-continuous cubic surface except at singularities



Comparing Catmull-Clark & Loop



Catmull-Clark



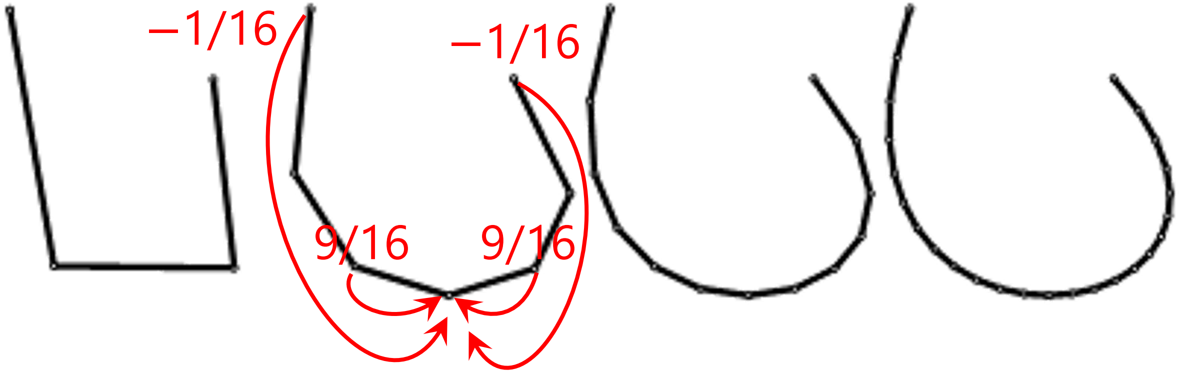
Loop

Catmull-Clark

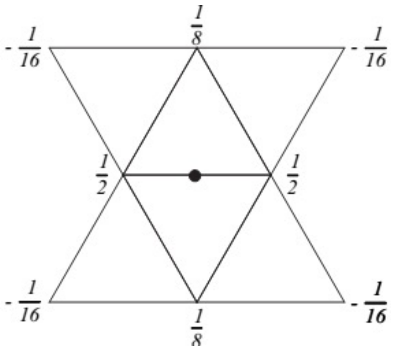
- Catmull-Clark is de facto standard in CG industry
  - Quad meshes can naturally represent two principal curvature directions

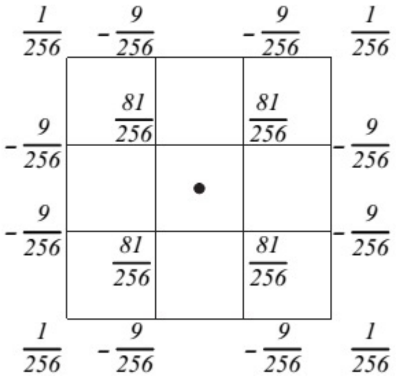
Other subdivision schemes

- Four-point method
  - Passes through CPs (interpolating)
    - ←→ approximating
  - Cannot be represented as polynomials
  - C^1-continous
  - Surface version: butterfly method



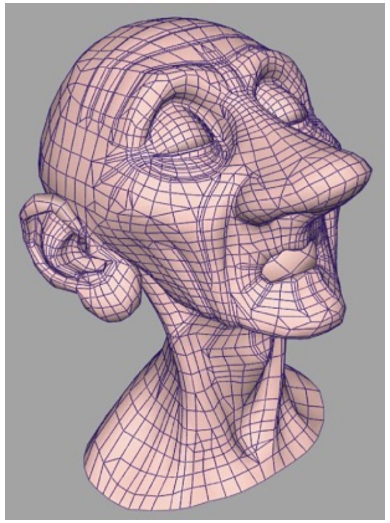
- Many more variants
  - Kobbelt's method
  - $\sqrt{3}$-method
  - etc...





Geri’s Game (Pixar, 1997)

- First film using subdivision surfaces
  - Previously (Toy Story), tedious modeling work using B-splines

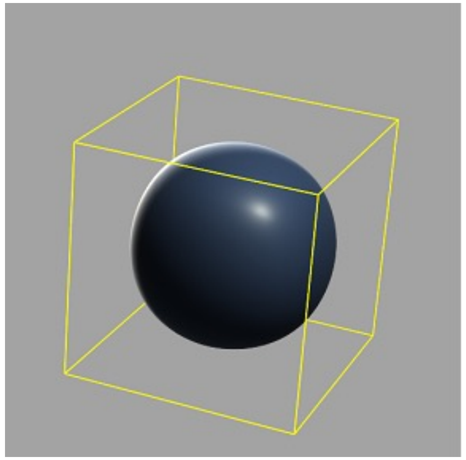


https://www.youtube.com/watch?v=9IYRC7g2ICg

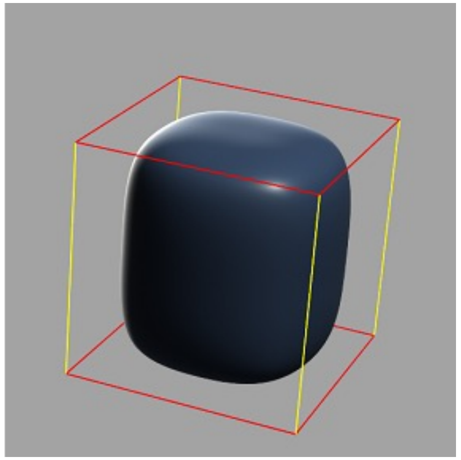
Subdivision surfaces in character animation [DeRose,Kass,Truong,SIGGRAPH98]

Controlling smoothness

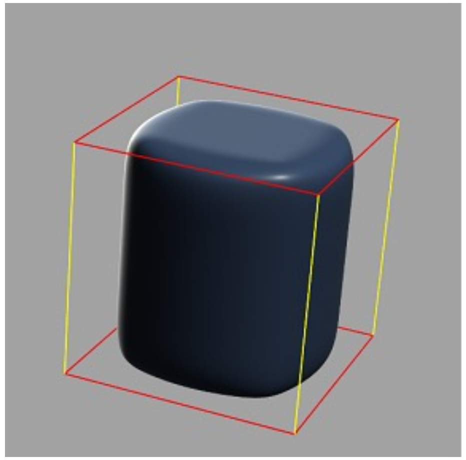
- Can represent sharp edges by altering subdivision rules



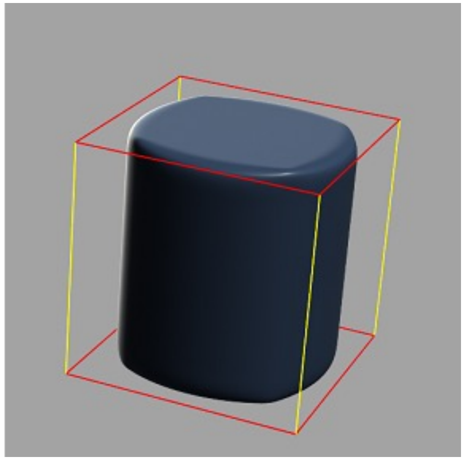
sharpness=0



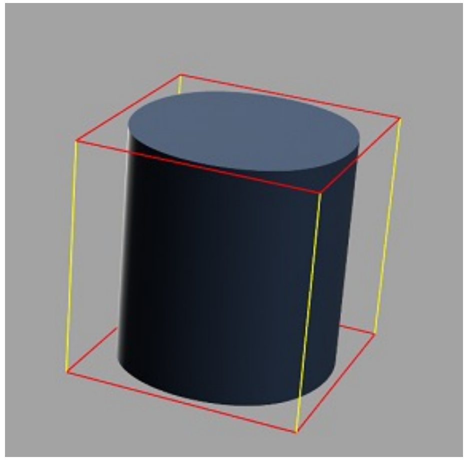
sharpness=1



sharpness=2



sharpness=3

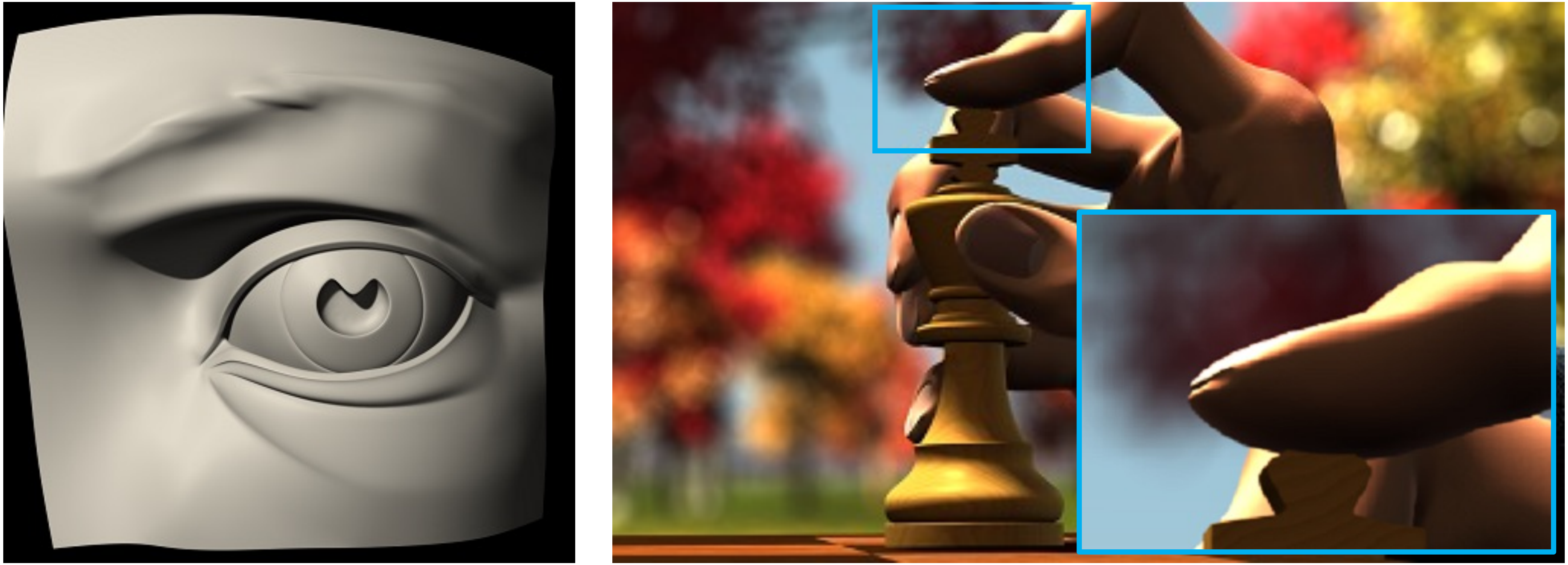


sharpness=∞

Subdivision surfaces in character animation [DeRose,Kass,Truong,SIGGRAPH98]

Controlling smoothness

- Can represent sharp edges by altering subdivision rules



Subdivision surfaces in character animation [DeRose,Kass,Truong,SIGGRAPH98]

Resources for learning subdivision techniques

- Smooth Subdivision Surfaces Based on Triangles [Loop, MSc. Thesis 87]
  - Thorough & visual explanation of literature (Doo-Sabin & Catmull-Clark)
  - Some known errors: http://www.cs.berkeley.edu/~sequin/CS284/TEXT/LoopErrata.txt

- Subdivision for Modeling and Animation [SIG00 Course]
  - Most famous survey, but a little arcane
  - http://www.cs.nyu.edu/~dzorin/sig00course/

- OpenSubdiv from research to industry adoption [SIG13 Course]
  - More recent topics
  - http://dx.doi.org/10.1145/2504435.2504451

Halfedge data structure

Mesh representation using vertex & face lists

OFF file format

```
OFF
8 6 0
-0.5 -0.5 0.5
0.5 -0.5 0.5
-0.5 0.5 0.5
0.5 0.5 0.5
-0.5 0.5 -0.5
0.5 0.5 -0.5
-0.5 -0.5 -0.5
0.5 -0.5 -0.5
4 0 1 3 2
4 2 3 5 4
4 4 5 7 6
4 6 7 1 0
4 1 7 5 3
4 6 0 2 4
```

Geometry data

Topology data

← #vertices, #faces

← xyz coord of 0th vertex

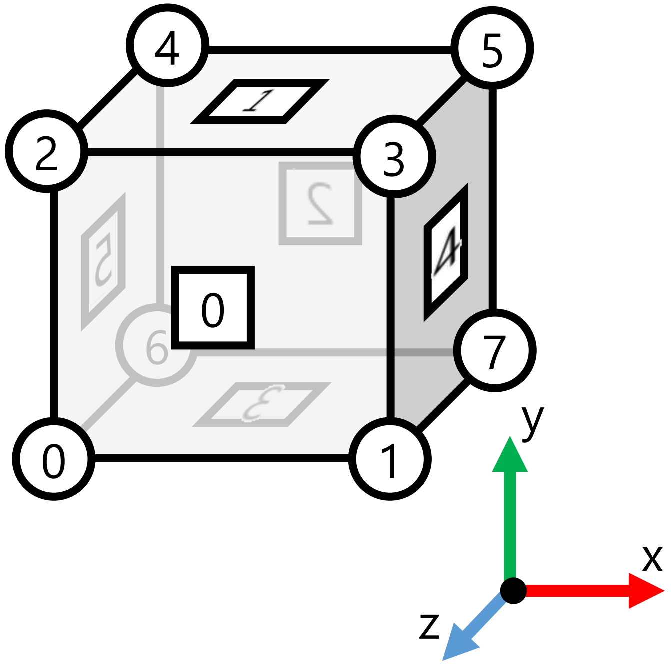
⋮

← xyz coord of 7th vertex

← 0th face's #vertices and vertex indices

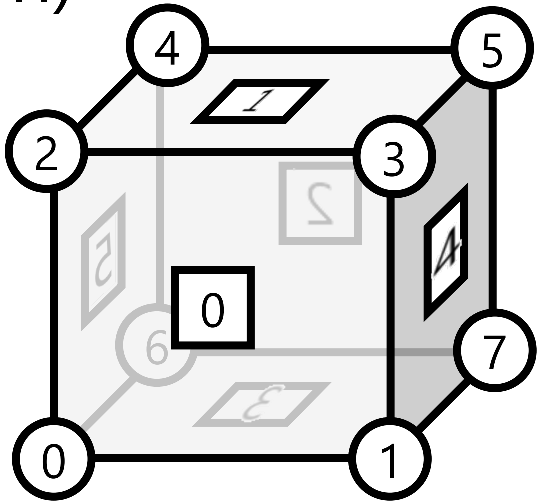
⋮

← 5th face's #vertices and vertex indices



Mesh representation using vertex & face lists

- Info needed during mesh processing (e.g. subdivision)
    - Set of faces around a vertex
    - Set of faces adjacent to a face
    - Vertices at an edge’s endpoints
    - Faces at both sides of an edge
    - etc…



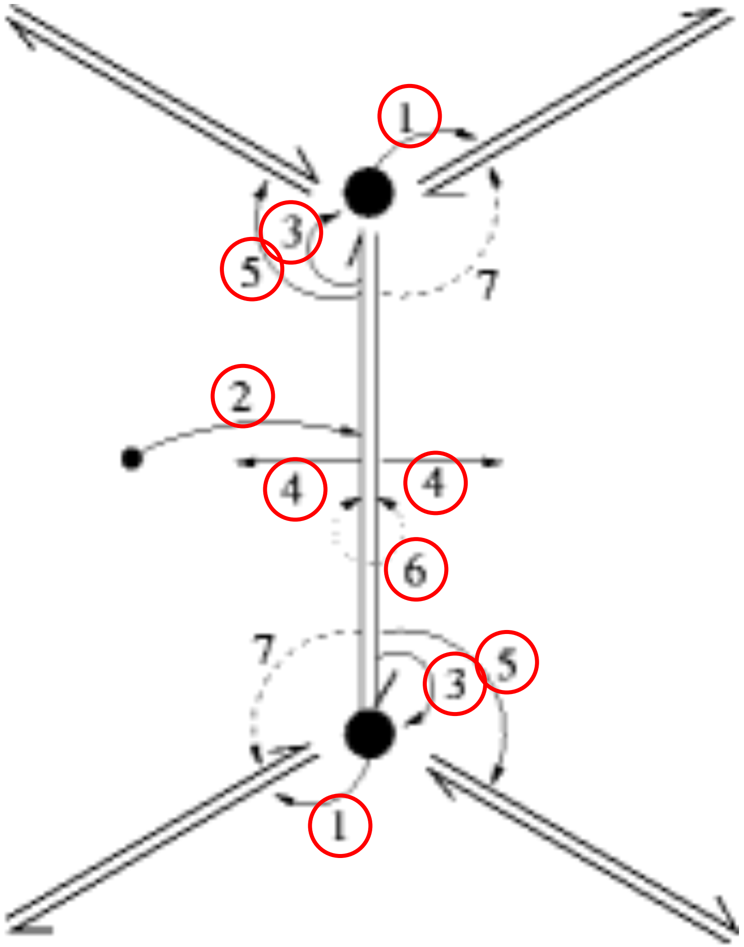
- Can be stored as “array or arrays”, but consumes more memory☹

Halfedge data structure

- Store link information:
  (1) Vertex → One of halfedges emanating from it
  (2) Face → One of halfedges composing it
  (3) Halfedge → Vertex that it points to
  (4) Halfedge → Face that it belongs to
  (5) Halfedge → Next halfedge
  (6) Halfedge → Halfedge opposite to it

- Circling around a face:
  (2) → (5) → (5) → …

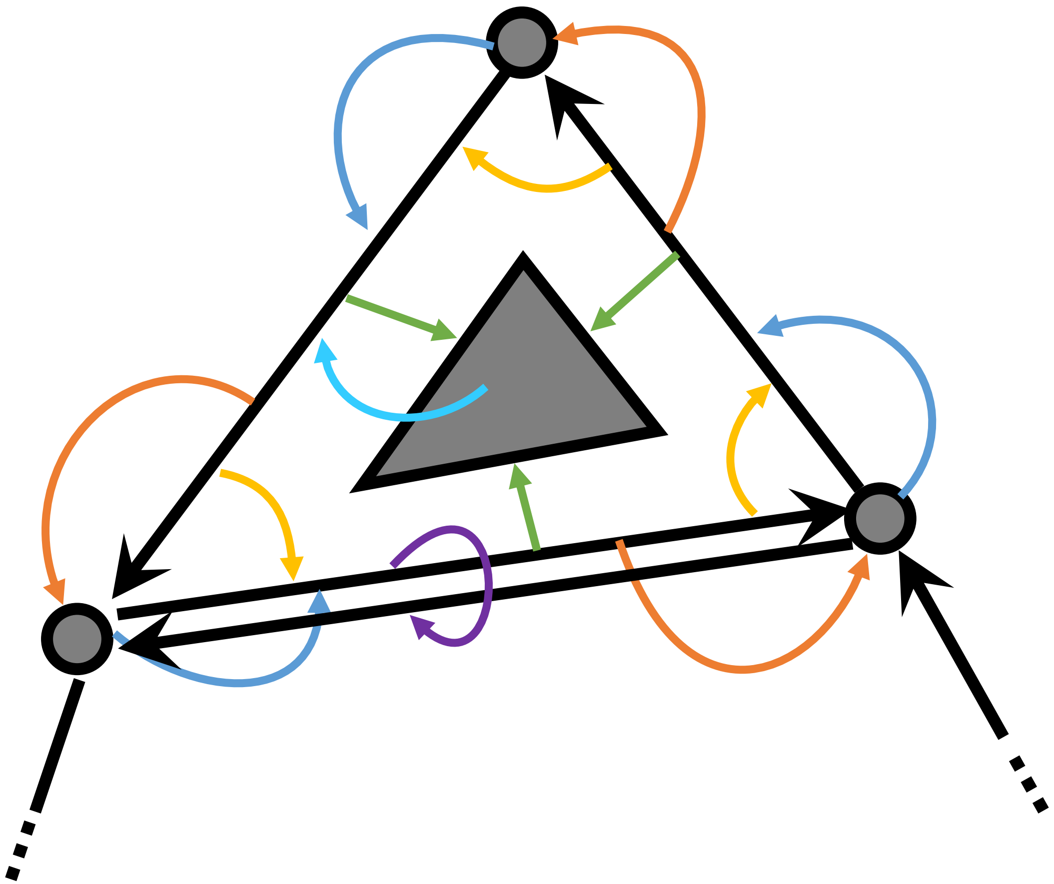
- Circling around a vertex:
  (1) → (6) → (5) → (6) → (5) → …



http://www.openmesh.org/

When a new face is added

- Generate halfedges
- Link vertex to halfedge (1)
- Link halfedge to vertex (3)
- Link halfedge to next halfedge (5)
- Link halfedges to face (4)
- Link face to halfedge (2)
- Link halfedge to its opposite halfedge if such exists (6)



Papers

- Recursively generated B-spline surfaces on arbitrary topological meshes [Catmull,Clark,CAD78]
- A 4-point interpolatory subdivision scheme for curve design [Dyn,Levin,CAGD87]
- A butterfly subdivision scheme for surface interpolation with tension control [Dyn,Levine,Gregory,TOG90]
- Sqrt(3)-subdivision [Kobbelt,SIGGRAPH00]
- Exact evaluation of Catmull-Clark subdivision surfaces at arbitrary parameter values [Stam,SIGGRAPH98]
- Interactive multiresolution mesh editing [Zorin,Schroder,Sweldens,SIGGRAPH97]
- Interpolating subdivision for meshes with arbitrary topology [Zorin,Schroder,Sweldens,SIGGRAPH96]