Introduction to Computer Graphics

– Modeling (1) –

April 15, 2021

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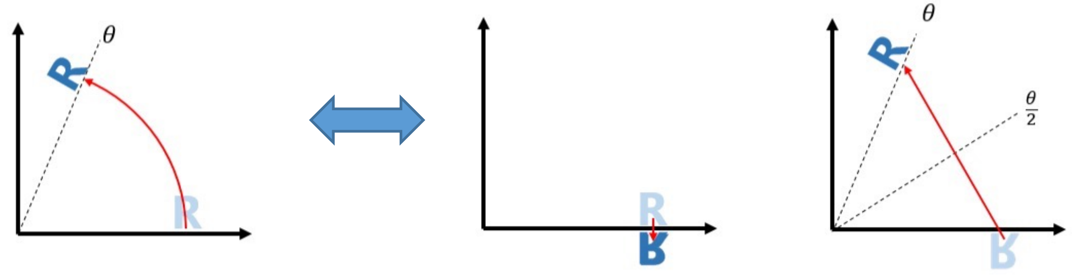
Some additional notes on quaternions

Another explanation for quaternions (overview)

1. Any rotation can be decomposed into even number of reflections



2. Quaternions can concisely describe reflections in 3D

$$R_{\vec{f}}(\vec{x}) = -\vec{f}\,\vec{x}\,\vec{f}^{-1}$$

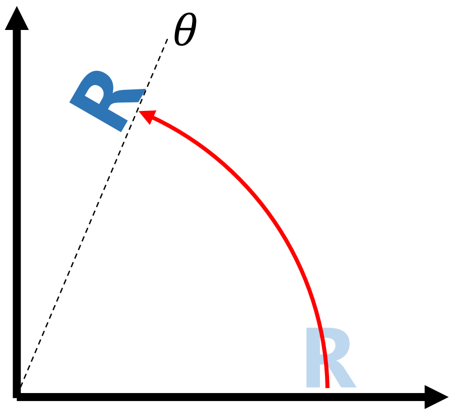
3. Combining two reflections equivalent to the rotation leads to the formula

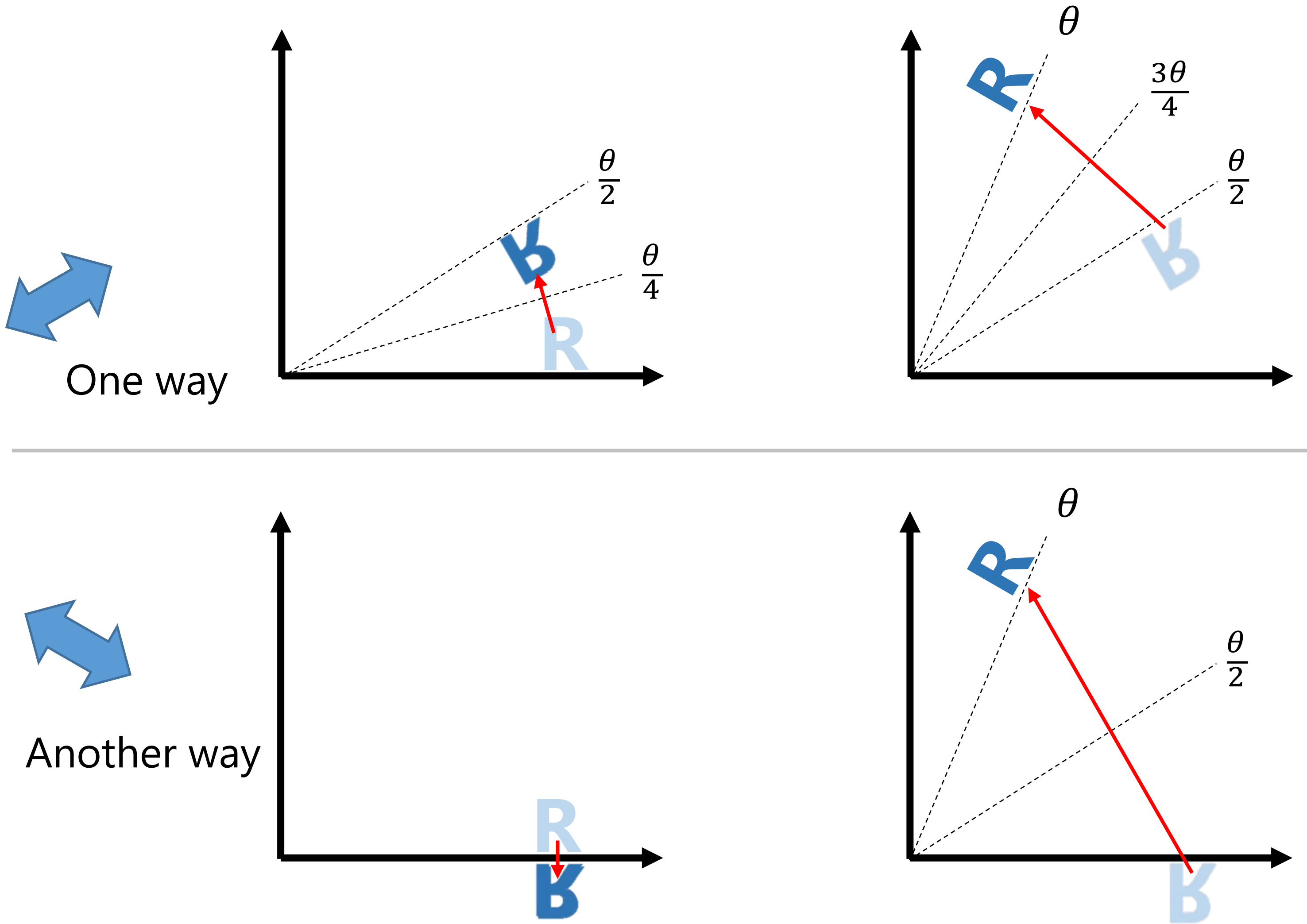
$$R_{\vec{g}}\left(R_{\vec{f}}(\vec{x})\right) = \left(\cos\frac{\theta}{2} + \vec{\omega}\sin\frac{\theta}{2}\right)\vec{x}\left(\cos\frac{\theta}{2} - \vec{\omega}\sin\frac{\theta}{2}\right)$$

Any rotation can be decomposed into even number of reflections

- Mathematically proven
  - Valid for any dimensions

- Not unique (of course!)





Quaternions recap

- Complex number: real + imaginary $\qquad a + b\,\mathbf{i}$

- Quaternion: scalar + vector $\qquad a + \vec{v}$
- Definition of quaternion multiplication:

$$(a_1 + \overrightarrow{v_1})(a_2 + \overrightarrow{v_2}) \quad \coloneqq \quad \overbrace{a_1 a_2 - \overrightarrow{v_1} \cdot \overrightarrow{v_2}}^{\text{Scalar part}} \quad + \quad \overbrace{a_1 \overrightarrow{v_2} + a_2 \overrightarrow{v_1} + \overrightarrow{v_1} \times \overrightarrow{v_2}}^{\text{Vector part}}$$

- Pure vectors can take multiplication by interpreting them as quaternions:

$$\overrightarrow{v_1}\,\overrightarrow{v_2} \quad = \quad -\overrightarrow{v_1} \cdot \overrightarrow{v_2} \quad + \quad \overrightarrow{v_1} \times \overrightarrow{v_2}$$

- Notable properties:
  - (Relevant later)

$$\boxed{\vec{v}\,\vec{v} = -\|\vec{v}\|^2}$$

$\vec{v} \times \vec{v}$ is always zero

$$\boxed{\vec{v}^{-1} = -\frac{\vec{v}}{\|\vec{v}\|^2}}$$

Multiplying $\vec{v}$ to rhs produces 1

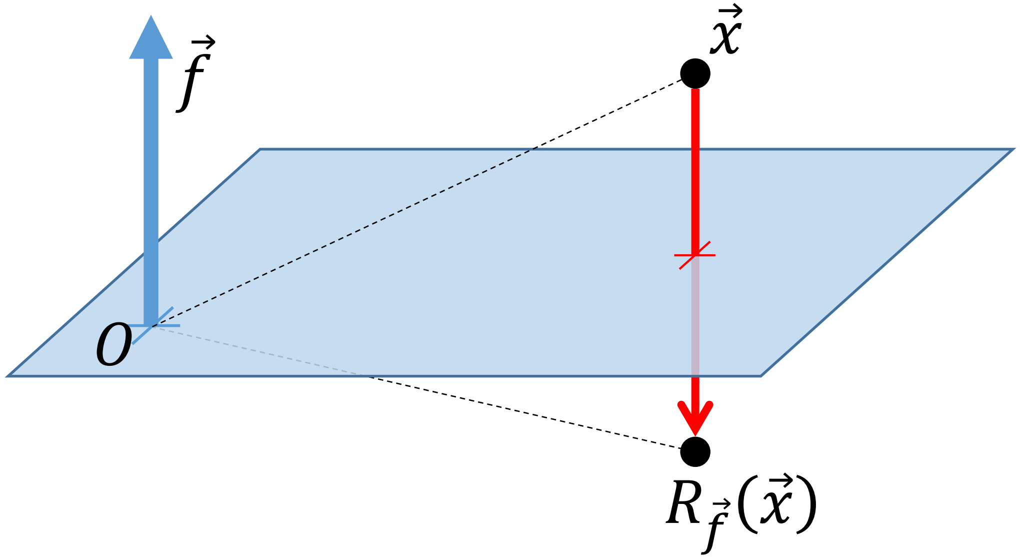
$$\boxed{\text{If } \vec{v} \cdot \vec{w} = 0\,, \text{ then } \vec{v}\,\vec{w} = -\vec{w}\,\vec{v}}$$

$\vec{v}\,\vec{w} = \vec{v} \times \vec{w} = -\vec{w} \times \vec{v} = -\vec{w}\,\vec{v}$

Describing reflections using quaternions

- Reflection of a point $\vec{x}$ across a plane orthogonal to $\vec{f}$:

$$R_{\vec{f}}(\vec{x}) \coloneqq -\vec{f}\,\vec{x}\,\vec{f}^{-1}$$



- Holds essential properties of reflections:
  - Linearity:

    $$R_{\vec{f}}(a\,\vec{x} + b\,\vec{y}) = a\,R_{\vec{f}}(\vec{x}) + b\,R_{\vec{f}}(\vec{y})$$

  - $\vec{f}$ gets mapped to $-\vec{f}$:

    $$R_{\vec{f}}(\vec{f}) = -\vec{f}\,\vec{f}\,\vec{f}^{-1} = -\vec{f}$$

  - If a point $\vec{x}$ satisfies $\vec{x} \cdot \vec{f} = 0$ (i.e. on the plane), $\vec{x}$ doesn't move:

    $$R_{\vec{f}}(\vec{x}) = -\vec{f}\,\vec{x}\,\vec{f}^{-1} = -\left(-\vec{x}\,\vec{f}\right)\vec{f}^{-1} = \vec{x}$$

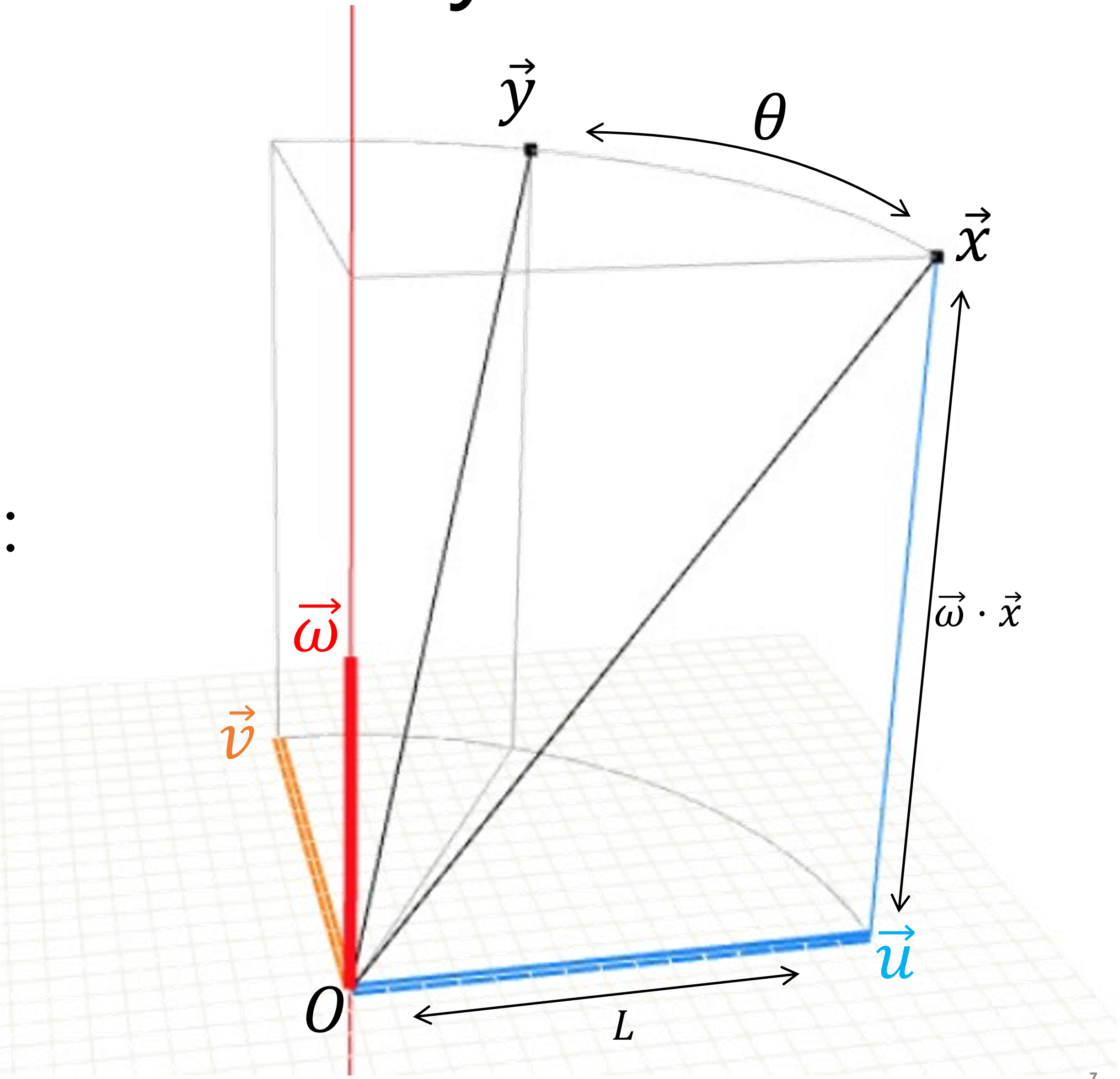
    Because if $\vec{x} \cdot \vec{f} = 0$, then $\vec{f}\,\vec{x} = -\vec{x}\,\vec{f}$

https://math.stackexchange.com/a/7263

Setup for rotation around arbitrary axis

- Rotation axis (unit vector)： $\vec{\omega}$
- Rotation angle： θ
- Point before rotation： $\vec{x}$
- Point after rotation： $\vec{y} \coloneqq R_{\vec{\omega},\theta}(\vec{x})$

- Think of local 2D coordinate system：
  - “Right” vector ： $\vec{u} \coloneqq \vec{x} - (\vec{\omega} \cdot \vec{x})\vec{\omega}$
  - “Up” vector： $\vec{v} \coloneqq \vec{\omega} \times \vec{x}$
    - Note that $\|\vec{u}\| = \|\vec{v}\|$
    - (Let’s call it L)

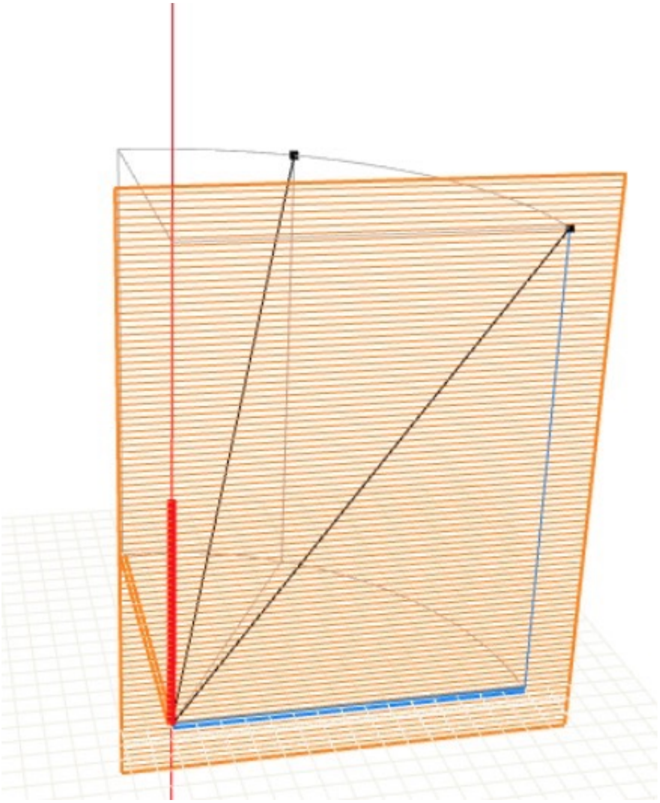


https://legacygl-js.glitch.me/demo/quaternion-schematic.html

Decompose rotation into two reflections

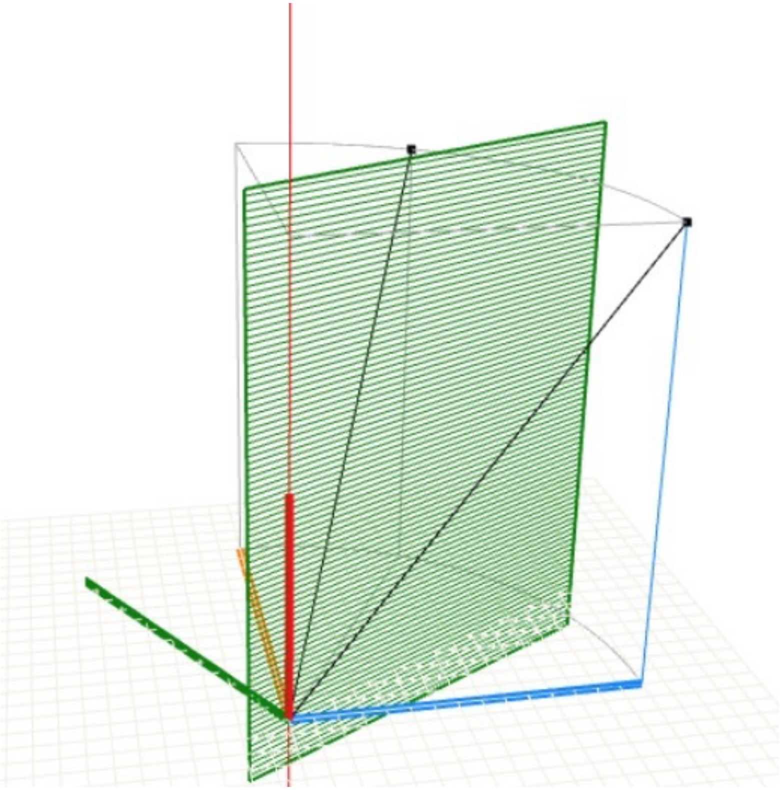
1st reflection：

$$\vec{f} \coloneqq \vec{v}$$

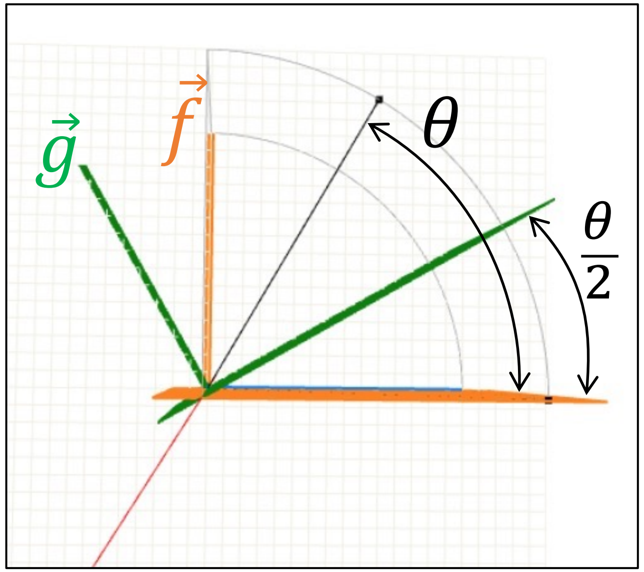


2nd reflection：

$$\vec{g} \coloneqq -\sin\frac{\theta}{2}\,\vec{u} + \cos\frac{\theta}{2}\,\vec{v}$$



Top view



https://legacygl-js.glitch.me/demo/quaternion-schematic.html

Combining two reflections

- Formula： $R_{\vec{g}}\left(R_{\vec{f}}(\vec{x})\right) = R_{\vec{g}}(-\vec{f}\,\vec{x}\,\vec{f}^{-1}) = -\vec{g}(-\vec{f}\,\vec{x}\,\vec{f}^{-1})\vec{g}^{-1} = (\vec{g}\,\vec{f})\,\vec{x}\,(\vec{f}^{-1}\,\vec{g}^{-1})$
  - Substitute $\vec{f} \coloneqq \vec{v}, \quad \vec{g} \coloneqq -\sin\frac{\theta}{2}\,\vec{u} + \cos\frac{\theta}{2}\,\vec{v}$ to the above
- For the left part $\vec{g}\,\vec{f}$ ：

$$\vec{g}\cdot\vec{f} = \left(-\sin\tfrac{\theta}{2}\,\vec{u} + \cos\tfrac{\theta}{2}\,\vec{v}\right)\cdot\vec{v} = L^2\cos\tfrac{\theta}{2}$$ (because $\vec{u}\cdot\vec{v} = 0$)

$$\vec{g}\times\vec{f} = \left(-\sin\tfrac{\theta}{2}\,\vec{u} + \cos\tfrac{\theta}{2}\,\vec{v}\right)\times\vec{v} = -L^2\sin\tfrac{\theta}{2}\,\vec{\omega}$$ (because $\vec{u}\times\vec{v} = L^2\vec{\omega}$)

  Therefore,

$$\vec{g}\,\vec{f} = -\vec{g}\cdot\vec{f} + \vec{g}\times\vec{f} = -L^2\left(\cos\tfrac{\theta}{2} + \vec{\omega}\,\sin\tfrac{\theta}{2}\right)$$

- The right part $\vec{f}^{-1}\,\vec{g}^{-1} = \frac{\vec{f}\,\vec{g}}{L^4}$ is analogous： $\vec{f}^{-1}\,\vec{g}^{-1} = -L^{-2}\left(\cos\frac{\theta}{2} - \vec{\omega}\,\sin\frac{\theta}{2}\right)$
- Finally, we get the formula：

$$R_{\vec{\omega},\theta}(\vec{x}) = R_{\vec{g}}\left(R_{\vec{f}}(\vec{x})\right) = \left(\cancel{-L}^2\left(\cos\tfrac{\theta}{2} + \vec{\omega}\,\sin\tfrac{\theta}{2}\right)\right)\vec{x}\left(\cancel{-L}^{-2}\left(\cos\tfrac{\theta}{2} - \vec{\omega}\,\sin\tfrac{\theta}{2}\right)\right)$$

$$= \boxed{\left(\cos\tfrac{\theta}{2} + \vec{\omega}\,\sin\tfrac{\theta}{2}\right)\vec{x}\left(\cos\tfrac{\theta}{2} - \vec{\omega}\,\sin\tfrac{\theta}{2}\right)}$$

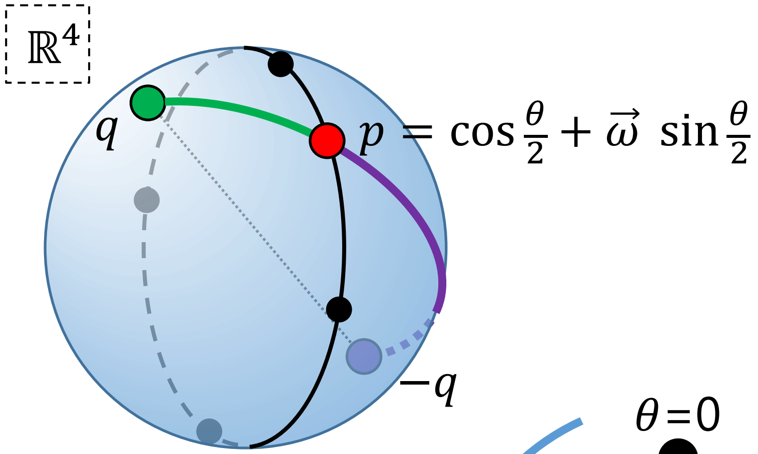
Representing and blending poses using quaternions

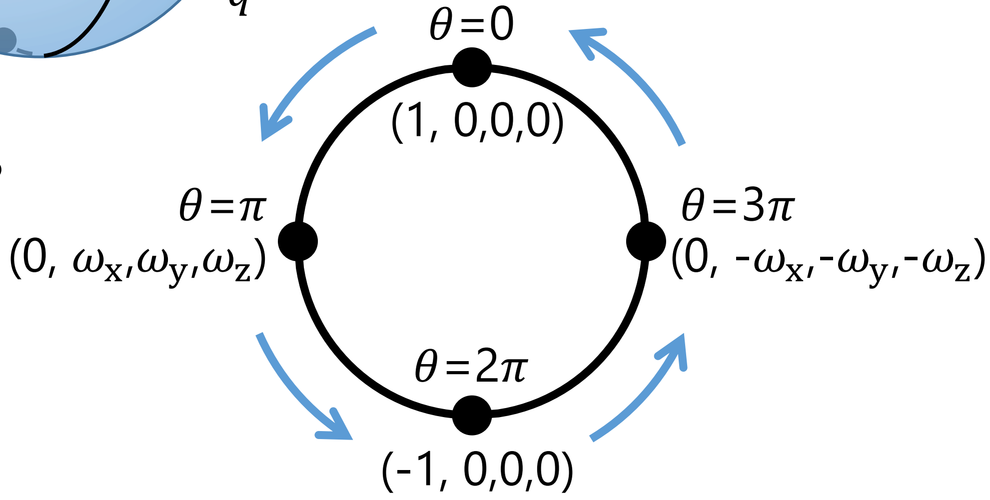
- Any rotations (poses) can be represented as unit quaternions
  - Points on hypersphere of 4D space

- Fix $\vec{\omega}$ and vary θ
  ➔ unit circle in 4D space

- A pose after rotating 360° about a certain axis is represented as another quaternion
  - One pose corresponds to two quaternions (double cover)

- A geodesic between two points p, q on the hypersphere represents interpolation of these poses
  - Should pick either q or $-q$ which is closer to p (i.e. 4D dot product is positive)





Normalize quaternions or not?

- Any quaternions can be written as scaling of unit quaternions

$$q = r\left(\cos\tfrac{\theta}{2} + \vec{\omega}\ \sin\tfrac{\theta}{2}\right), \qquad q^{-1} = r^{-1}\left(\cos\tfrac{\theta}{2} - \vec{\omega}\ \sin\tfrac{\theta}{2}\right)$$

- In the rotation formula, the scaling part is cancelled

$$q\ \vec{x}\ q^{-1} = \cancel{r}\left(\cos\tfrac{\theta}{2} + \vec{\omega}\ \sin\tfrac{\theta}{2}\right)\vec{x}\ \cancel{r}^{-1}\left(\cos\tfrac{\theta}{2} - \vec{\omega}\ \sin\tfrac{\theta}{2}\right) = \left(\cos\tfrac{\theta}{2} + \vec{\omega}\ \sin\tfrac{\theta}{2}\right)\vec{x}\left(\cos\tfrac{\theta}{2} - \vec{\omega}\ \sin\tfrac{\theta}{2}\right)$$

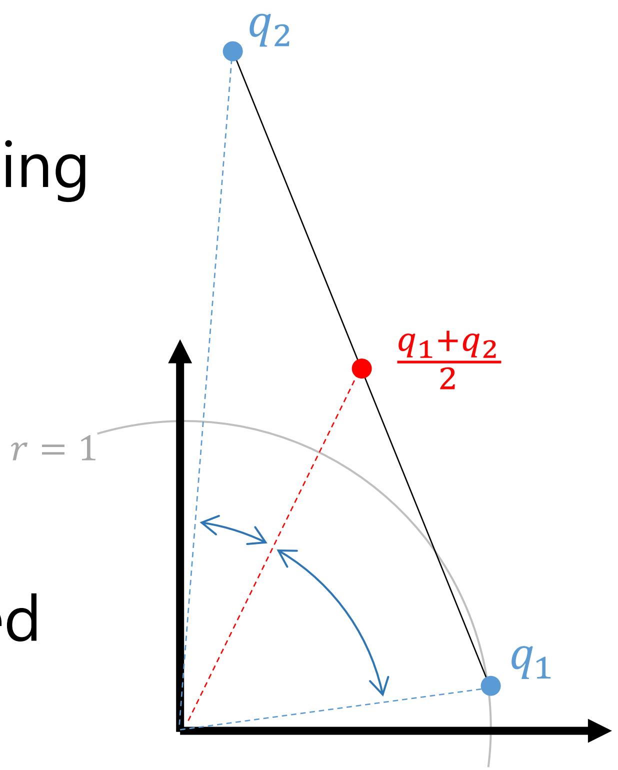
➔ so, normalization isn't needed?

- In practice, don't use quaternion mults for computing coordinate transformation (because inefficient)
    - Just do explicit vector calc using axis & angle

$$(\vec{x} - (\vec{\omega}\cdot\vec{x})\vec{\omega})\cos\theta + (\vec{\omega}\times\vec{x})\sin\theta + (\vec{\omega}\cdot\vec{x})\vec{\omega}$$

    - Can get axis & angle only after normalization

- Un-normalized can cause artifact when interpolated

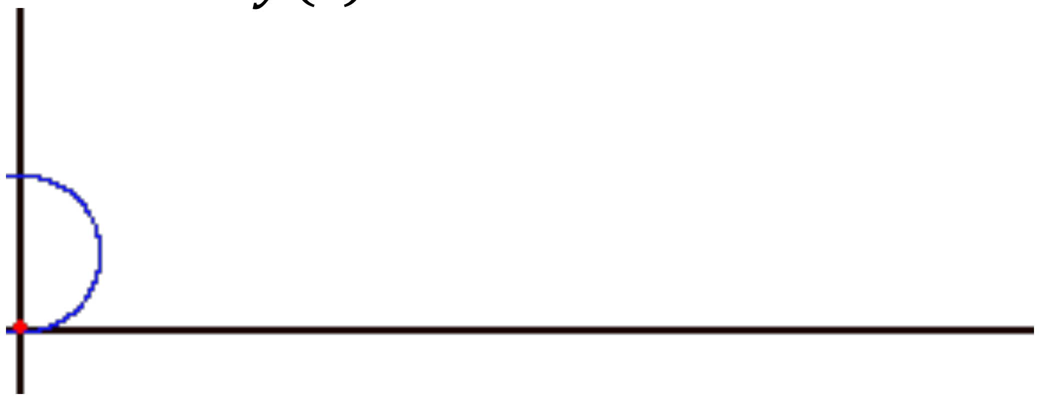


Modeling curves

Parametric curves

- X & Y coordinates defined by parameter t (≅ time)
    - Example: Cycloid

$$x(t) = t - \sin t$$
$$y(t) = 1 - \cos t$$



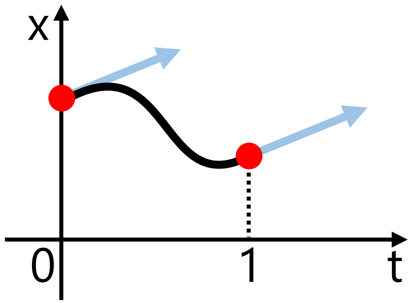
- Tangent (aka. derivative, gradient) vector: $\left(x'(t), y'(t)\right)$

- Polynomial curve: $x(t) = \sum_i a_i t^i$

Cubic Hermite curves

- Cubic polynomial curve interpolating derivative constraints at both ends (Hermite interpolation)

$$x(0) = x_0$$
$$x(1) = x_1$$
$$x'(0) = x'_0$$
$$x'(1) = x'_1$$



- 4 constraints ➔ 4 DoF needed ➔ 4 coefficients ➔ cubic
  - $x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$
  - $x'(t) = a_1 + 2a_2 t + 3a_3 t^2$

$$\begin{aligned} x(0) &= a_0 &&= x_0 \\ x(1) &= a_0 + a_1 + a_2 + a_3 &&= x_1 \\ x'(0) &= a_1 &&= x'_0 \\ x'(1) &= a_1 + 2\,a_2 + 3\,a_3 &&= x'_1 \end{aligned}$$

➔

$$\begin{aligned} a_0 &= x_0 \\ a_1 &= x'_0 \\ a_2 &= -3\,x_0 + 3\,x_1 - 2\,x'_0 - x'_1 \\ a_3 &= 2\,x_0 - 2\,x_1 + x'_0 + x'_1 \end{aligned}$$

- Coeffs determined by substituting constrained values & derivatives

Bezier curves

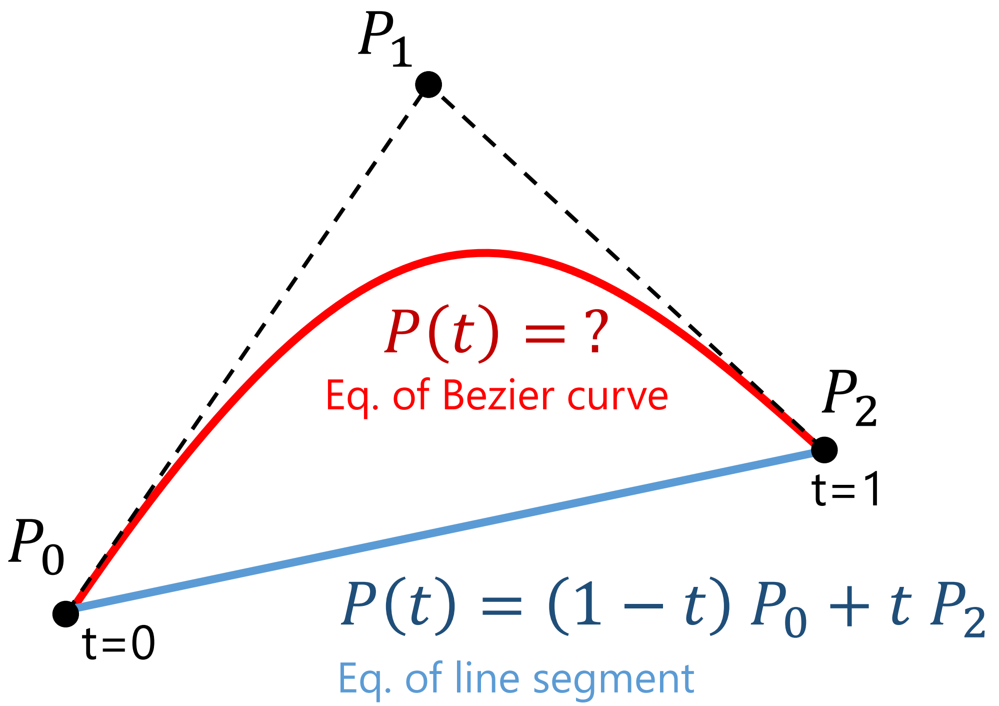
- Input: 3 control points (CPs) P_0, P_1, P_2
  - Coordinates of points in arbitrary domain (2D, 3D, …)

- Output: Curve $P(t)$ satisfying

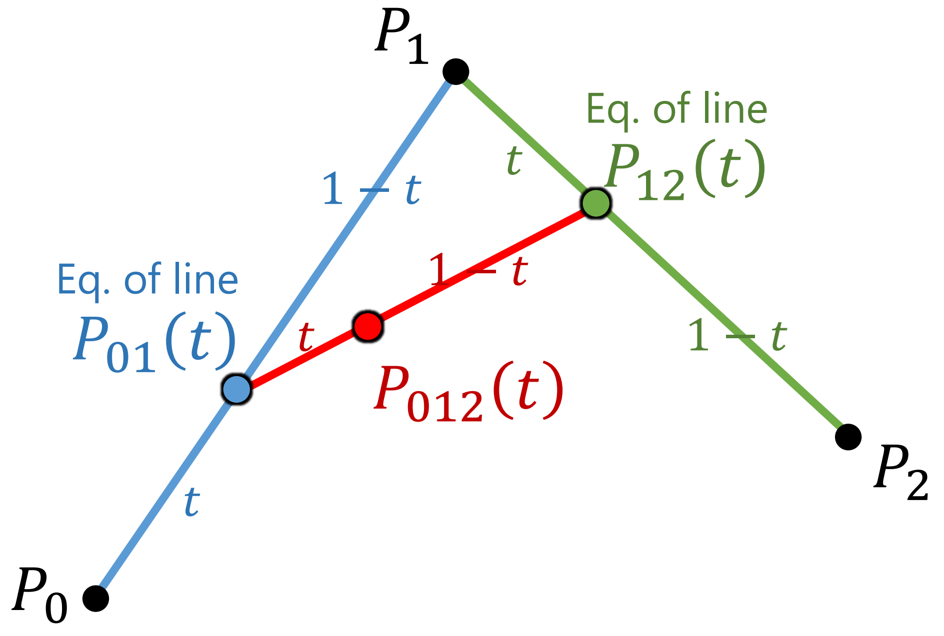
  $P(0) = P_0$

  $P(1) = P_2$

  while being "pulled" by P_1



Bezier curves



- $P_{01}(t) = (1-t)P_0 + t\,P_1$
- $P_{12}(t) = (1-t)P_1 + t\,P_2$
  - $P_{01}(0) = P_0$
  - $P_{12}(1) = P_2$

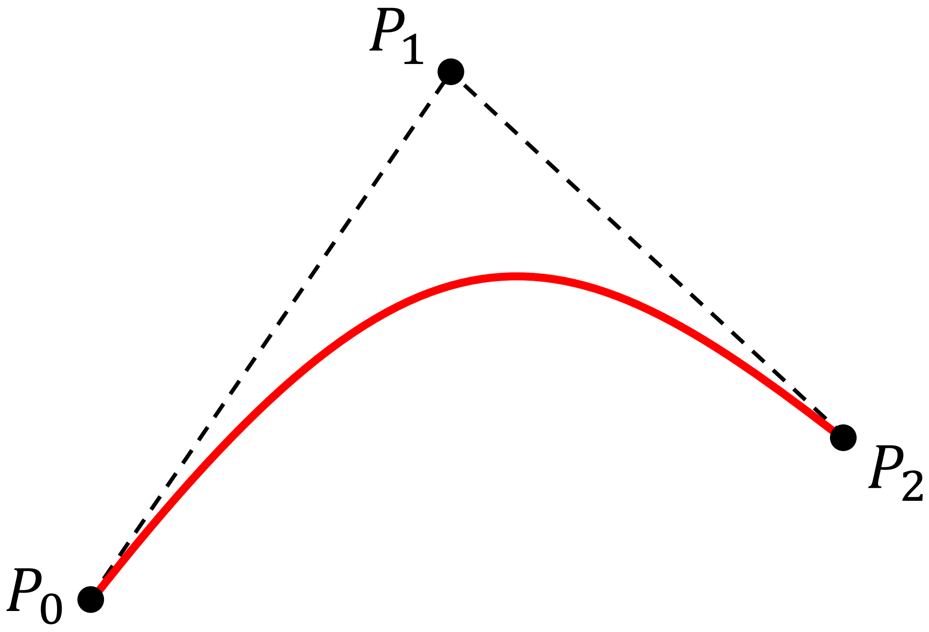
- Idea: "Interpolate the interpolation"
  As t changes $0 \to 1$, smoothly transition from P_{01} to P_{12}

- $P_{012}(t) = (1-t)P_{01}(t) + t\,P_{12}(t)$

$$= (1-t)\{(1-t)P_0 + t\,P_1\} + t\,\{(1-t)P_1 + t\,P_2\}$$

$$= \underbrace{(1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2}_{\text{Quadratic Bezier curve}}$$

Bezier curves



- $P_{01}(t) = (1-t)P_0 + t\,P_1$
- $P_{12}(t) = (1-t)P_1 + t\,P_2$
  - $P_{01}(0) = P_0$
  - $P_{12}(1) = P_2$

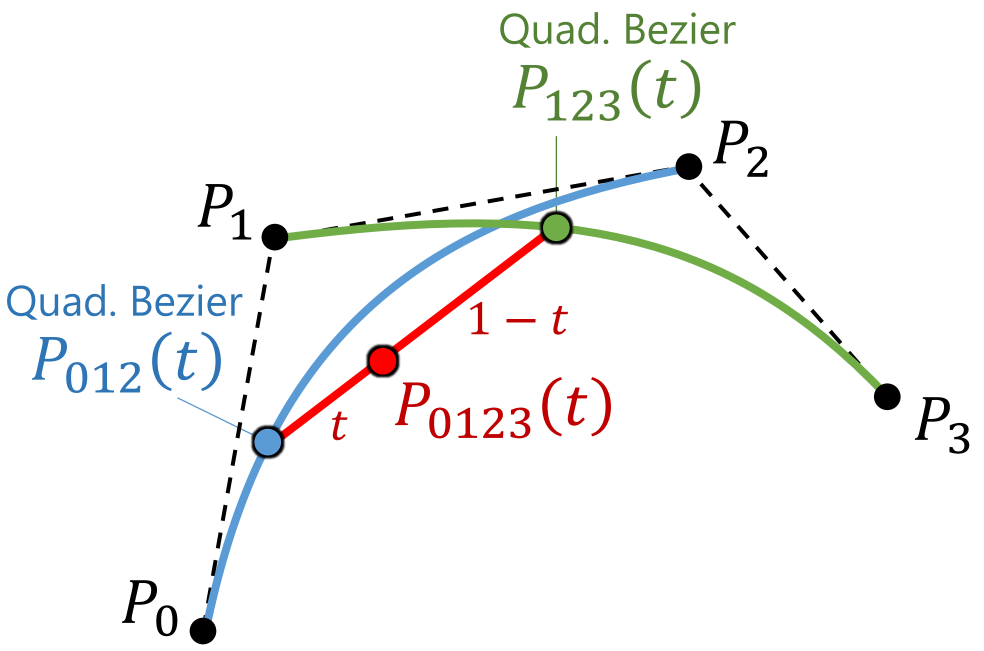
- Idea: "Interpolate the interpolation"
  As t changes $0 \rightarrow 1$, smoothly transition from P_{01} to P_{12}

- $P_{012}(t) = (1-t)P_{01}(t) + t\,P_{12}(t)$

$$= (1-t)\{(1-t)P_0 + t\,P_1\} + t\,\{(1-t)P_1 + t\,P_2\}$$

$$= \underbrace{(1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2}_{\text{Quadratic Bezier curve}}$$

Cubic Bezier curve



- Exact same idea applied to 4 points $P_0, P_1, P_2\ P_3$:
  - As t changes $0 \rightarrow 1$, transition from P_{012} to P_{123}

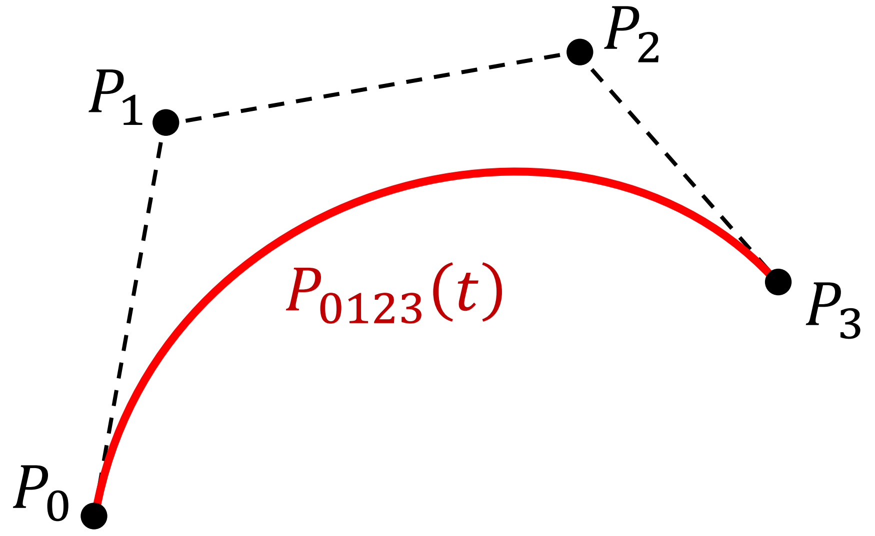
- $P_{0123}(t) = (1-t)P_{012}(t) + t\ P_{123}(t)$

$$= (1-t)\{(1-t)^2P_0 + 2t(1-t)P_1 + t^2P_2\} + t\,\{(1-t)^2P_1 + 2t(1-t)P_2 + t^2P_3\}$$

$$= \underbrace{(1-t)^3P_0 + 3t(1-t)^2P_1 + 3t^2(1-t)P_2 + t^3P_3}_{\text{Cubic Bezier curve}}$$

Cubic Bezier curve

- Exact same idea applied to 4 points $P_0, P_1, P_2\ P_3$:
  - As t changes $0 \rightarrow 1$, transition from P_{012} to P_{123}



- $P_{0123}(t) = (1-t)P_{012}(t) + t\ P_{123}(t)$

$$= (1-t)\{(1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2\} + t\,\{(1-t)^2 P_1 + 2t(1-t)P_2 + t^2 P_3\}$$

$$= \underbrace{(1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3}_{\text{Cubic Bezier curve}}$$

- Can easily control tangent at endpoints ➔ ubiquitously used in CG

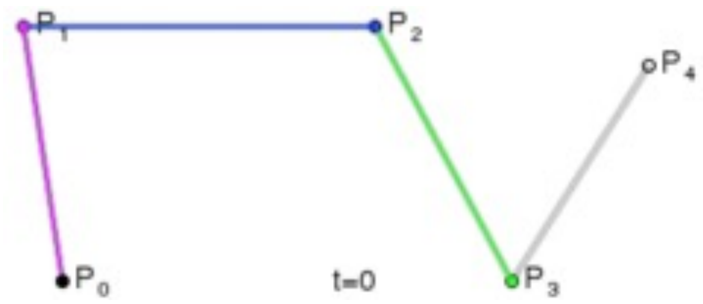
n-th order Bezier curve

- Input: n+1 control points $P_0, \cdots, P_n$

$$P(t) = \sum_{i=0}^{n} \underbrace{{}_n\mathrm{C}_i\, t^i\, (1-t)^{n-i}}_{b_i^n(t)}\, P_i$$

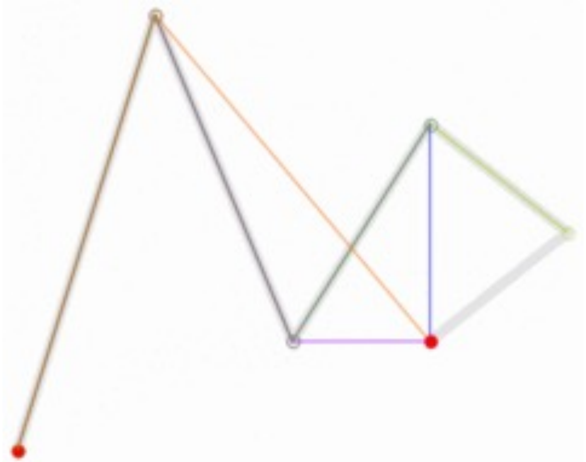
Bernstein basis function

Quartic (4th)



$$(1-t)^4 P_0 \quad +$$
$$4t(1-t)^3 P_1 \quad +$$
$$6t^2(1-t)^2 P_2 \quad +$$
$$4t^3(1-t) P_3 \quad +$$
$$t^4 P_4$$

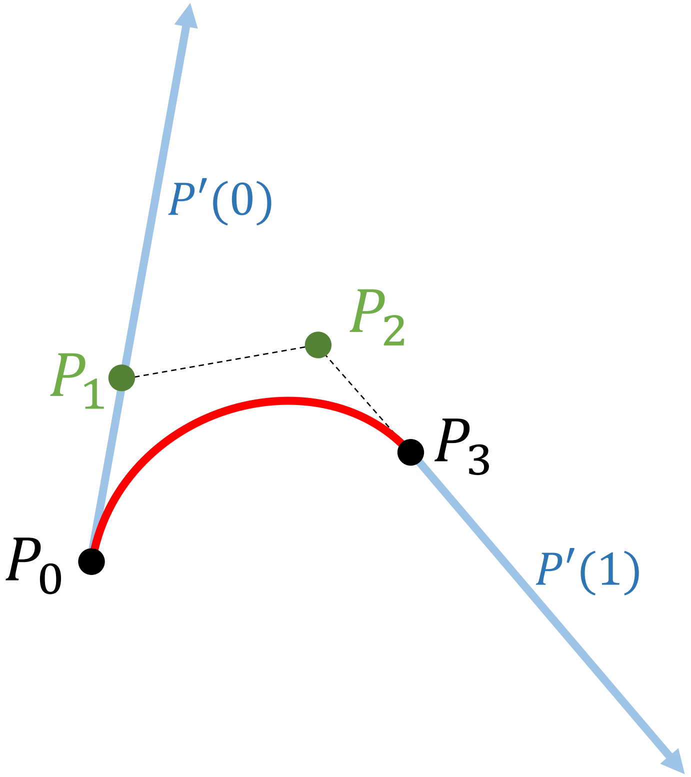
Quintic (5th)



$$(1-t)^5 P_0 \quad +$$
$$5t(1-t)^4 P_1 \quad +$$
$$10t^2(1-t)^3 P_2 \quad +$$
$$10t^3(1-t)^2 P_3 \quad +$$
$$5t^4(1-t) P_4 \quad +$$
$$t^5 P_5$$

Cubic Bezier curves & cubic Hermite curves

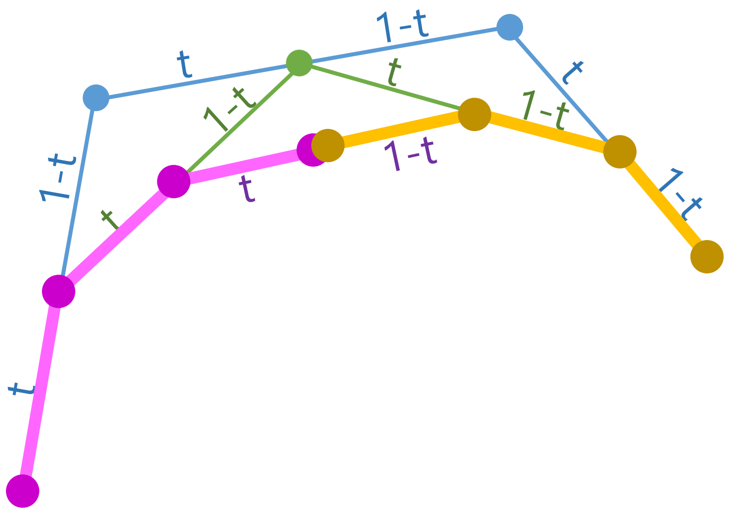
- Cubic Bezier curve & its derivative:
  - $P(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t) P_2 + t^3 P_3$
  - $P'(t) = -3(1-t)^2 P_0 + 3\{(1-t)^2 - 2t(1-t)\} P_1 + 3\{2t(1-t) - t^2\} P_2 + 3t^2 P_3$
- Derivatives at endpoints:
  - $P'(0) = -3P_0 + 3P_1$ ➔ $P_1 = P_0 + \frac{1}{3} P'(0)$
  - $P'(1) = -3P_2 + 3P_3$ ➔ $P_2 = P_3 - \frac{1}{3} P'(1)$
- Different ways of looking at cubic curves, essentially the same



Evaluating Bezier curves

- Method 1: Direct evaluation of polynomials
  - Simple & fast☺, could be numerically unstable☹

- Method 2: de Casteljau's algorithm
  - Directly after the recursive definition of Bezier curves
  - More computation steps☹, numerically stable☺
  - Also useful for splitting Bezier curves

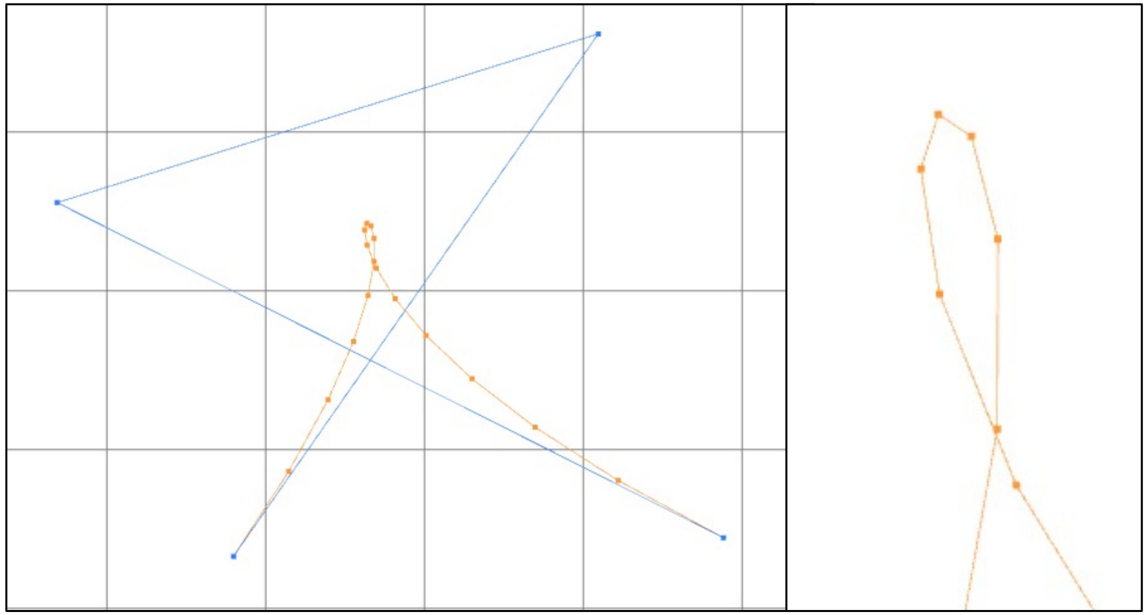


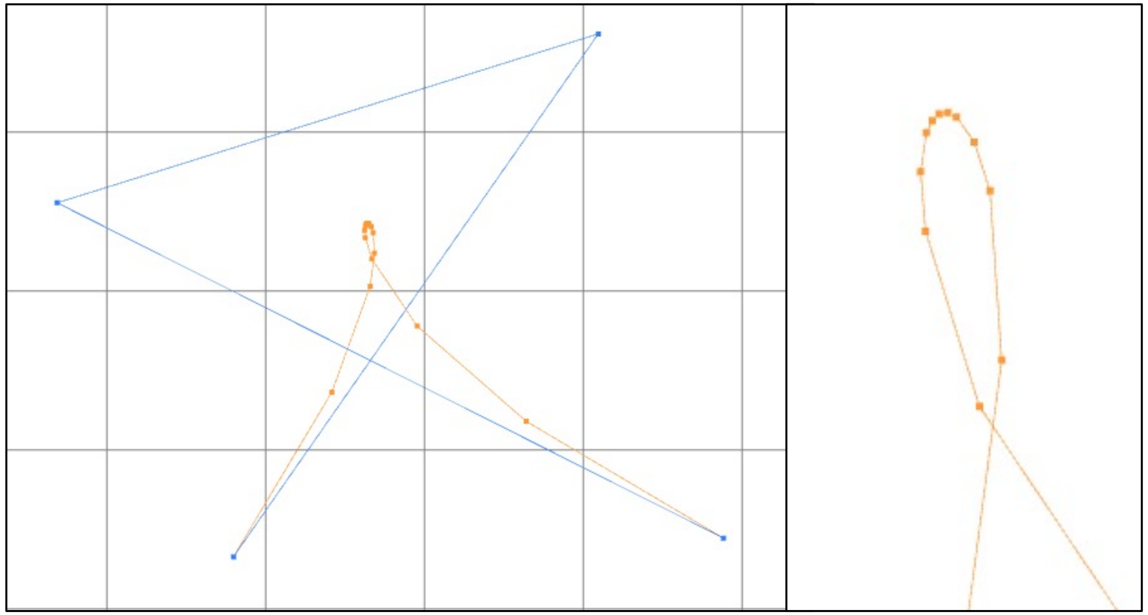
Drawing Bezier curves

- In the end, everything is drawn as polyline
  - Main question: How to sample parameter t?

- Method 1: Uniform sampling
  - Simple
  - Potentially insufficient sampling density

- Method 2: Adaptive sampling
  - If control points deviate too much from straight line, split by de Casteljau's algorithm





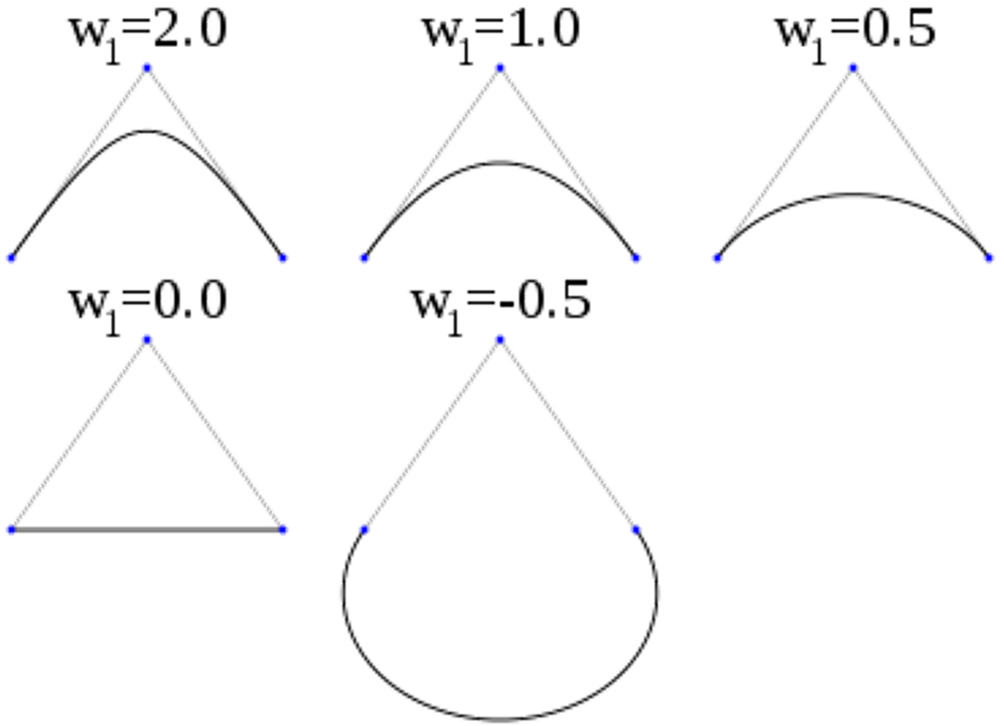
Further control: Rational Bezier curve

- Another view on Bezier curve:
  "Weighted average" of control points
  - $P_{012}(t) = (1-t)^2 P_0 + 2t(1-t) P_1 + t^2 \quad P_2$
    $= \lambda_0(t) \quad P_0 + \lambda_1(t) \quad P_1 + \lambda_2(t) P_2$
  - Important property: **partition of unity**
    $\lambda_0(t) + \lambda_1(t) + \lambda_2(t) = 1 \qquad \forall t$

- Multiply each $\lambda_i(t)$ by arbitrary coeff w_i:
  $$\xi_i(t) = w_i \, \lambda_i(t)$$

- Normalize to obtain new weights:
  $$\lambda'_i(t) = \frac{\xi_i(t)}{\sum_j \xi_j(t)}$$

$w_0 = w_2 = 1$



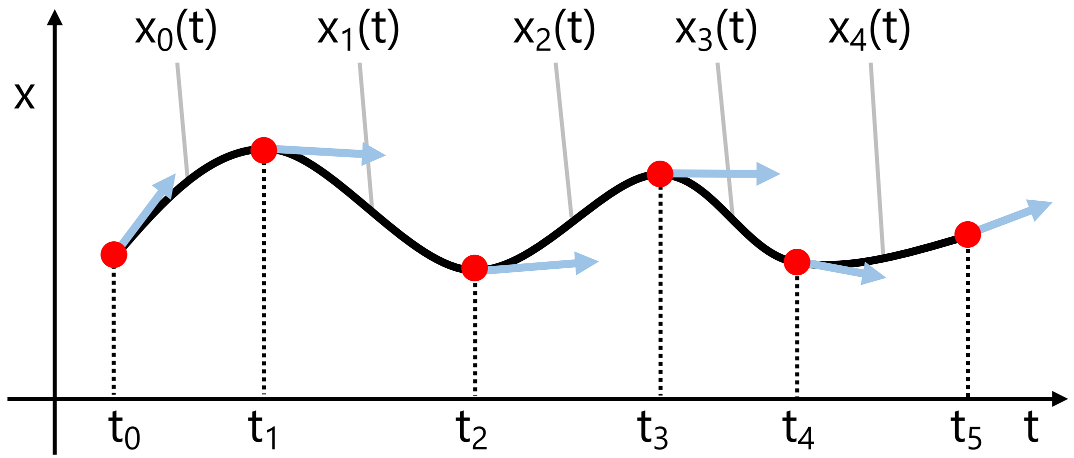
Non-polynomial curve ➔ can represent arcs etc.

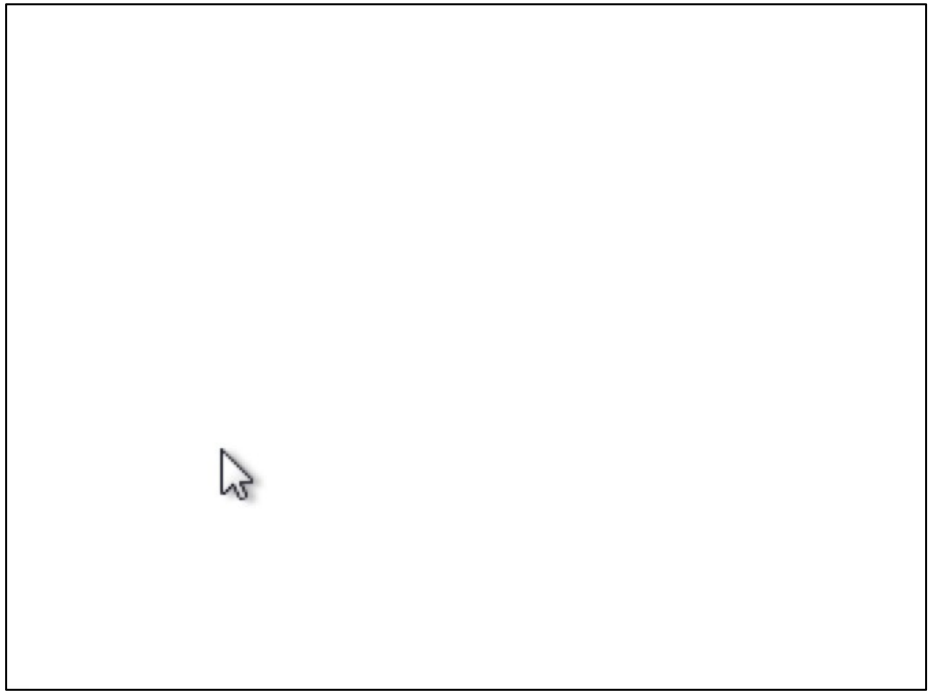
Cubic splines

- Series of connected cubic curves
  - Piecewise polynomial
  - Share value & derivative at every transition of intervals (C^1 continuity)

- Parameter range can be other than $[0, 1]$
  - Assumption: $t_k < t_{k+1}$

- Given values as only input,
  we want to automatically set derivatives

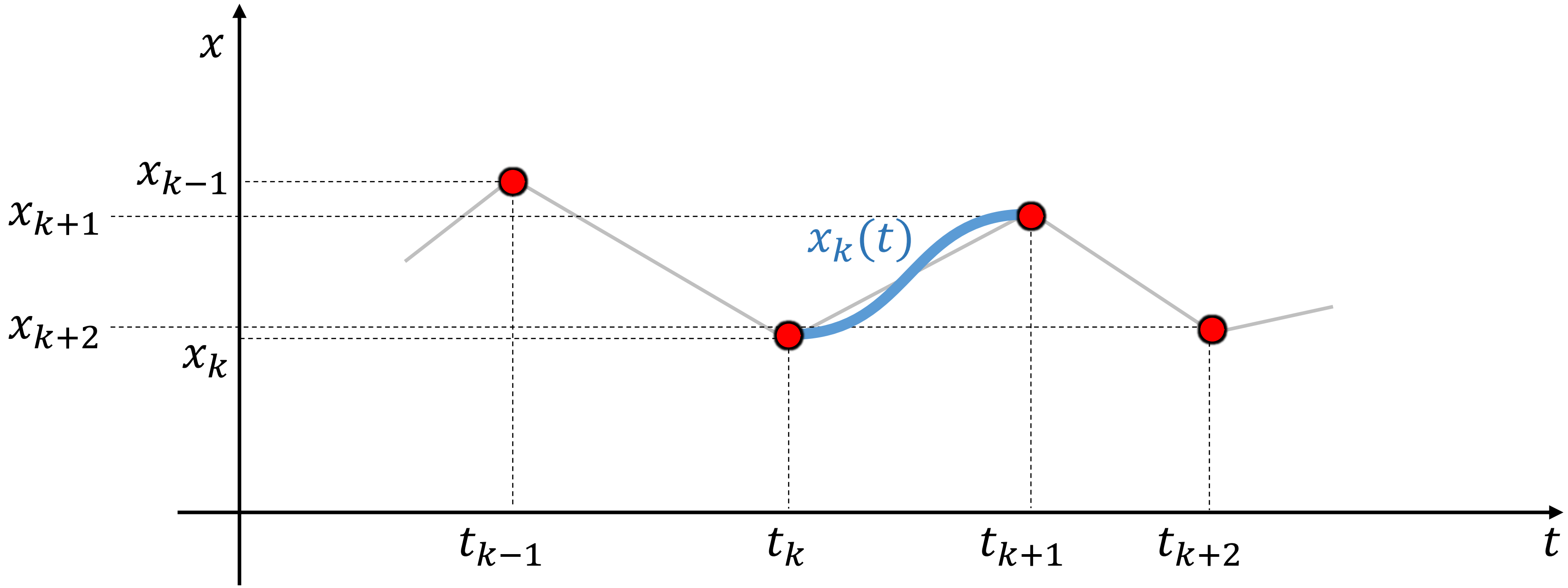




Curve tool in PowerPoint

Cubic Catmull-Rom spline

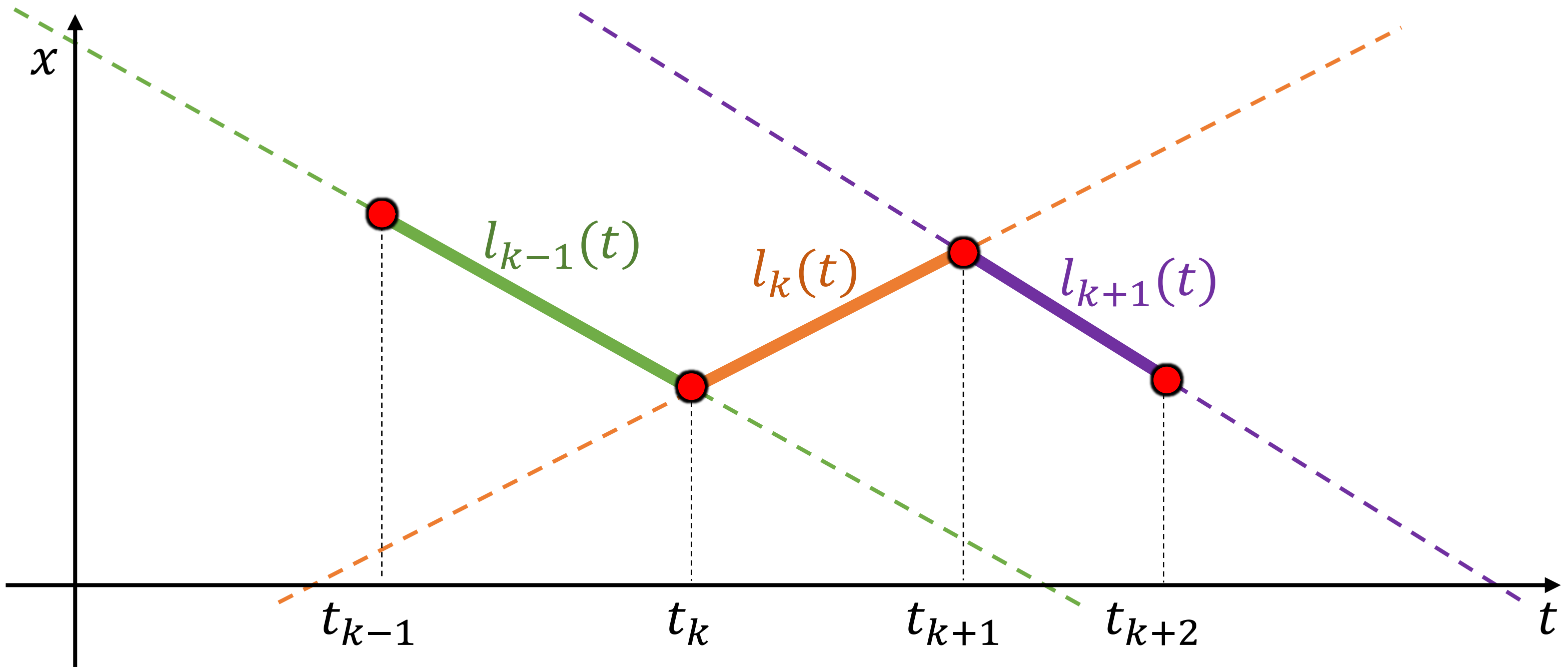
- Cubic function $x_k(t)$ for range $t_k \leq t \leq t_{k+1}$ is defined by adjacent constrained values x_{k-1}, x_k, x_{k+1}, x_{k+2}



Cubic Catmull-Rom spline: Step 1

- As $t_k \rightarrow t_{k+1}$, interpolate such that $x_k \rightarrow x_{k+1}$ ➔ Line

$$l_k(t) = \left(1 - \frac{t\ - t_k}{t_{k+1} - t_k}\right) x_k + \frac{t\ - t_k}{t_{k+1} - t_k} x_{k+1}$$

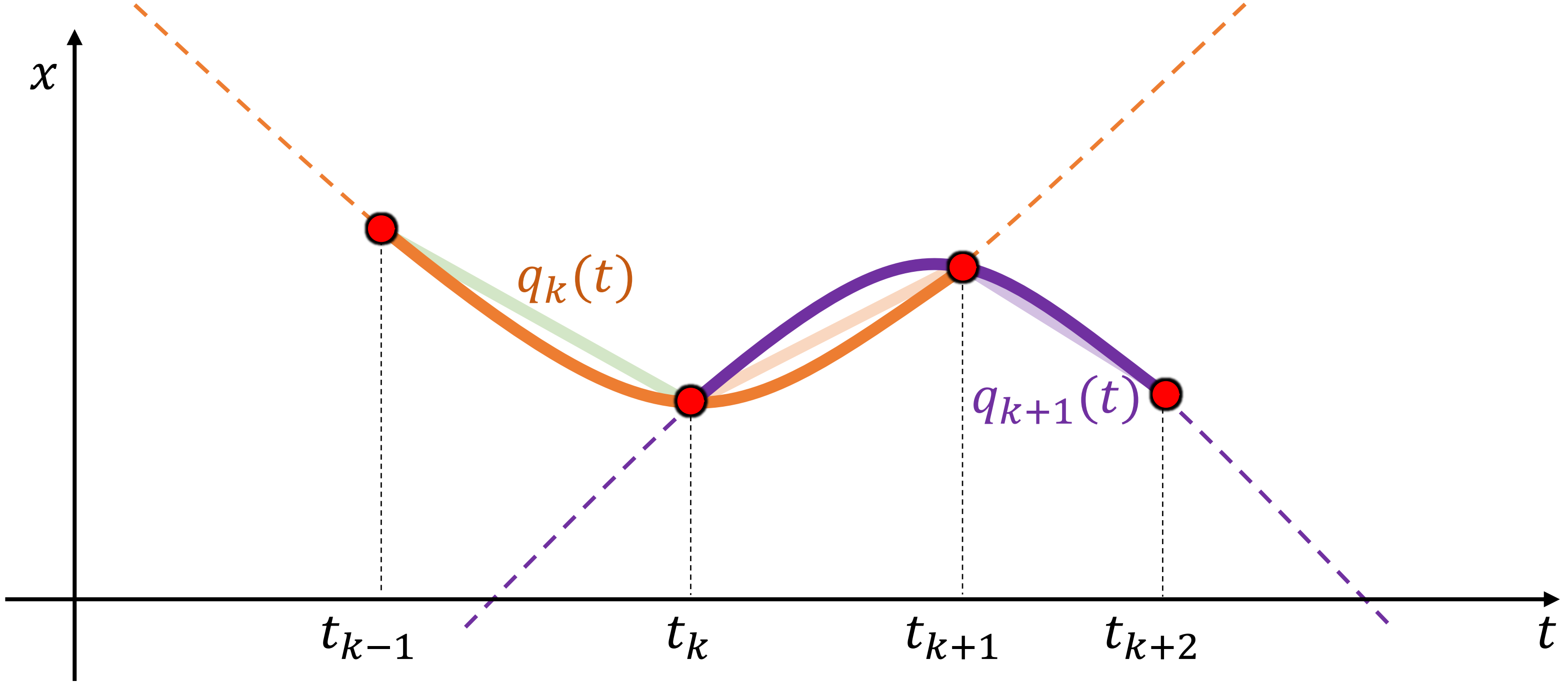


Cubic Catmull-Rom spline: Step 2

- As $t_{k-1} \rightarrow t_{k+1}$, interpolate such that $l_{k-1} \rightarrow l_k$ ➔ Quadratic curve

$$q_k(t) = \left(1 - \frac{t \; - t_{k-1}}{t_{k+1} - t_{k-1}}\right) l_{k-1}(t) + \frac{t \; - t_{k-1}}{t_{k+1} - t_{k-1}} l_k(t)$$

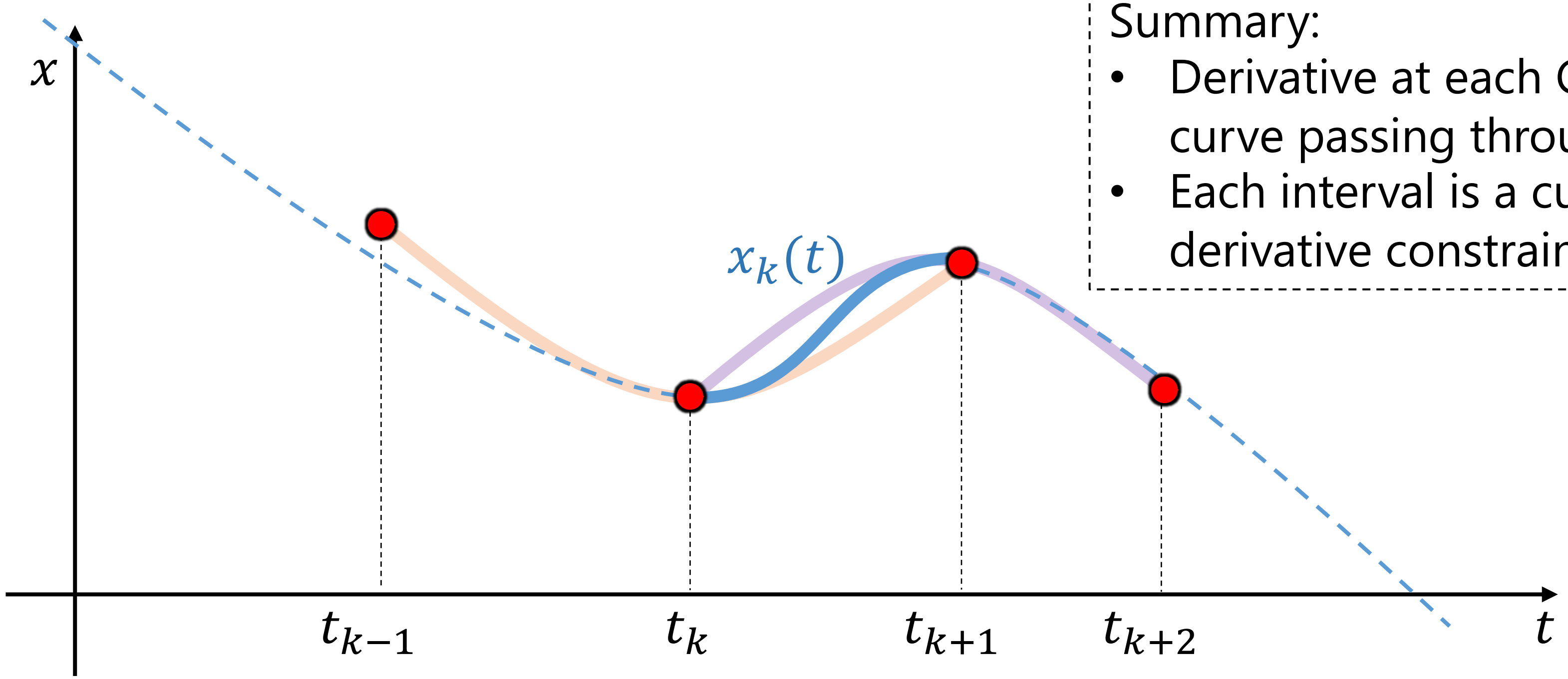
  - Passes through 3 points $(t_{k-1}, x_{k-1}), (t_k, x_k), (t_{k+1}, x_{k+1})$



Cubic Catmull-Rom spline: Step 3

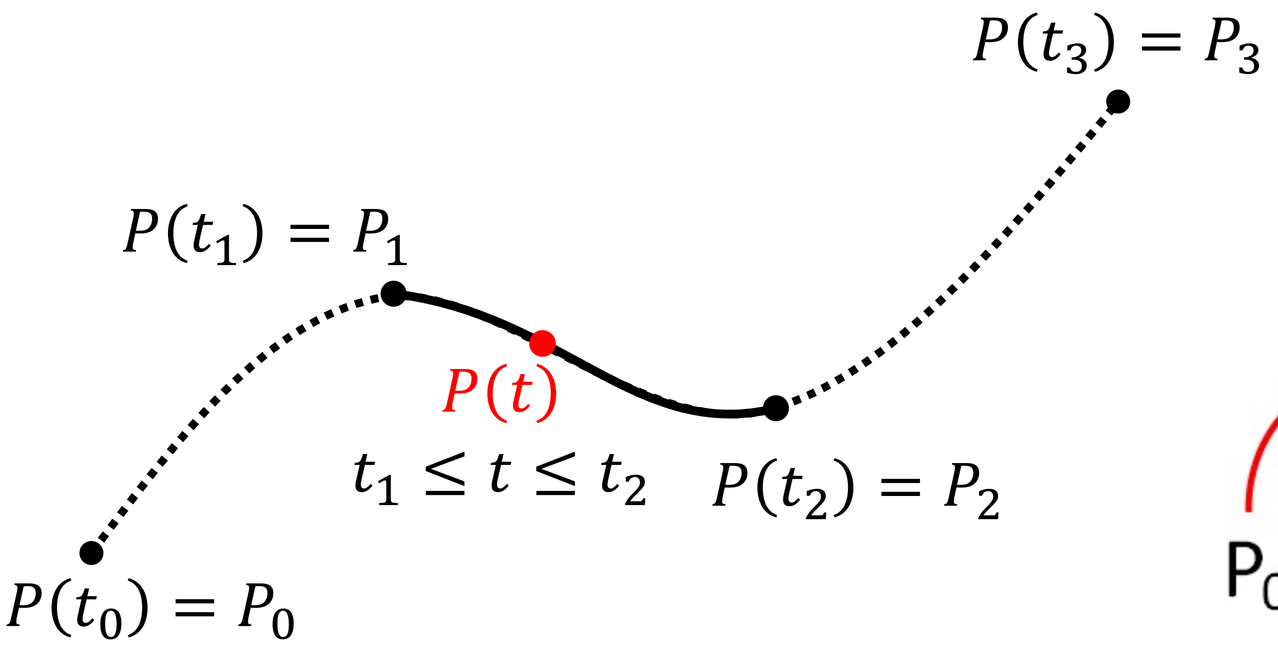
- As $t_k \to t_{k+1}$, interpolate such that $q_k \to q_{k+1}$ ➔ Cubic curve

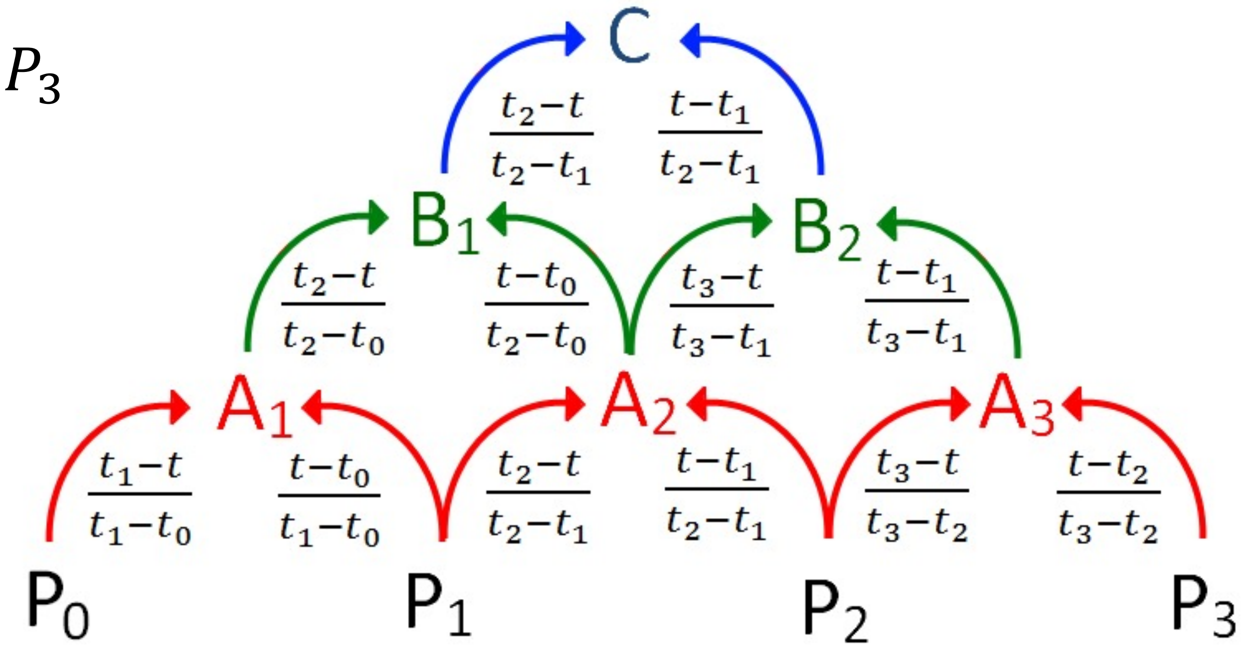
$$x_k(t) = \left(1 - \frac{t - t_k}{t_{k+1} - t_k}\right) q_k(t) + \frac{t - t_k}{t_{k+1} - t_k} q_{k+1}(t)$$



Summary:
- Derivative at each CP is defined by a quadratic curve passing through its adjacent CPs
- Each interval is a cubic curve satisfying derivative constraints at both ends

Evaluating cubic Catmull-Rom spline

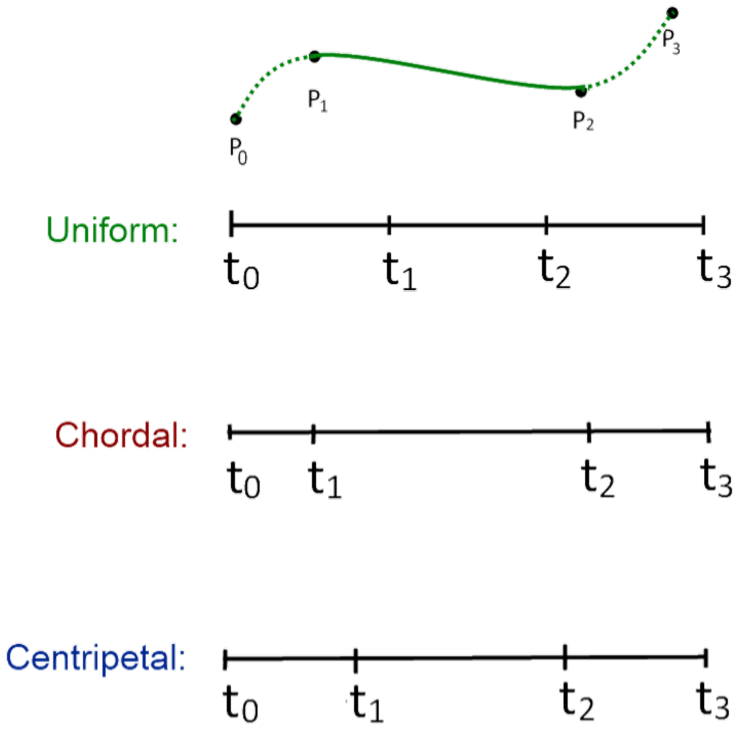


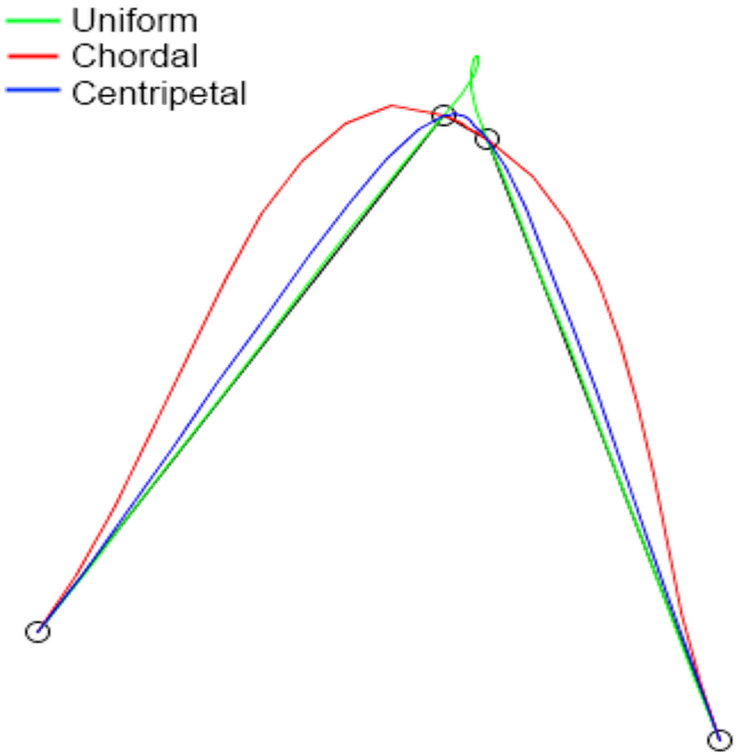


A recursive evaluation algorithm for a class of Catmull-Rom splines [Barry,Colgman,SIGGRAPH88]

Ways of setting parameter values t_k (aka. knot sequence)

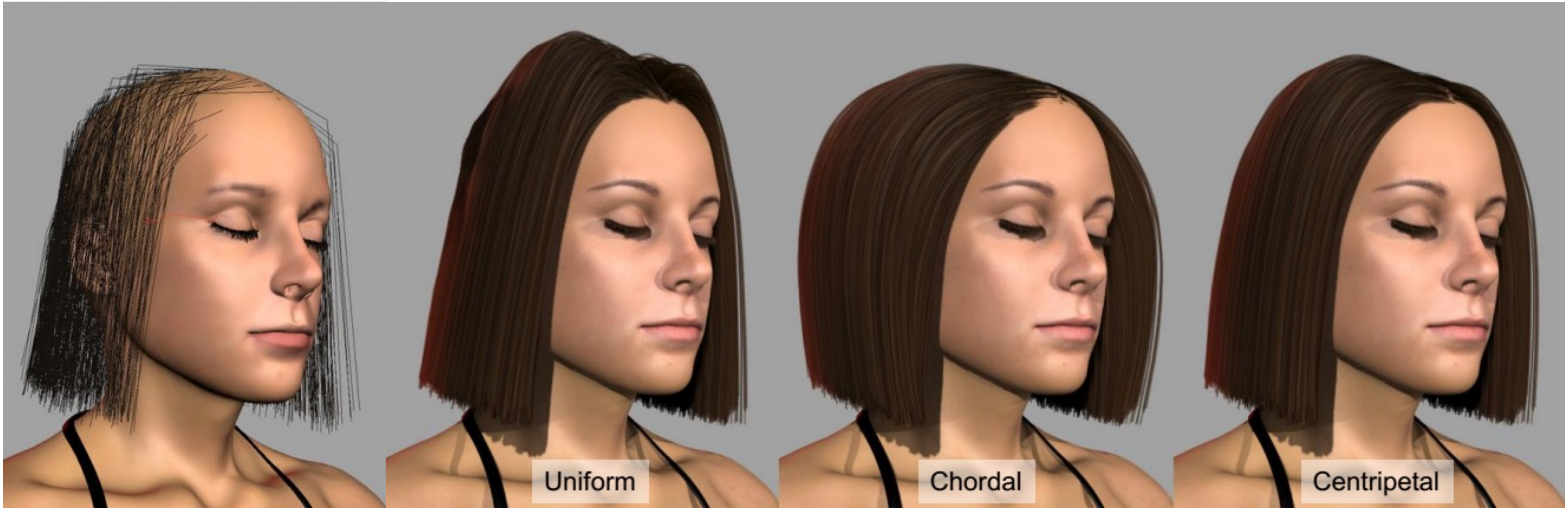
- Assume: $t_0 = 0$
- Uniform
  $$t_k = t_{k-1} + 1$$
- Chordal
  $$t_k = t_{k-1} + |P_{k-1} - P_k|$$
- Centripetal
  $$t_k = t_{k-1} + \sqrt{|P_{k-1} - P_k|}$$





Parameterization of Catmull-Rom Curves [Yuksel,Schaefer,Keyser,CAD11]

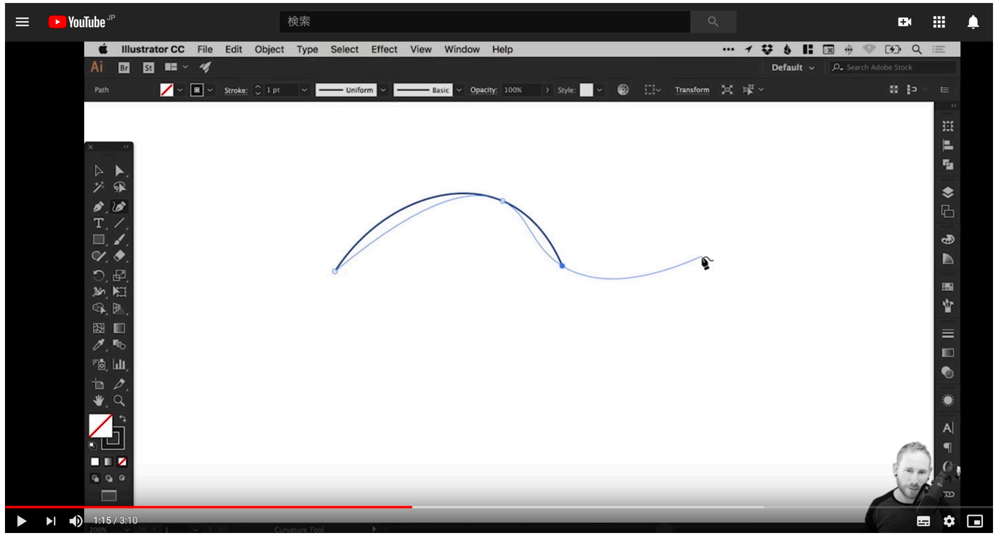
Application of cubic Catmull-Rom spline: Hair modeling



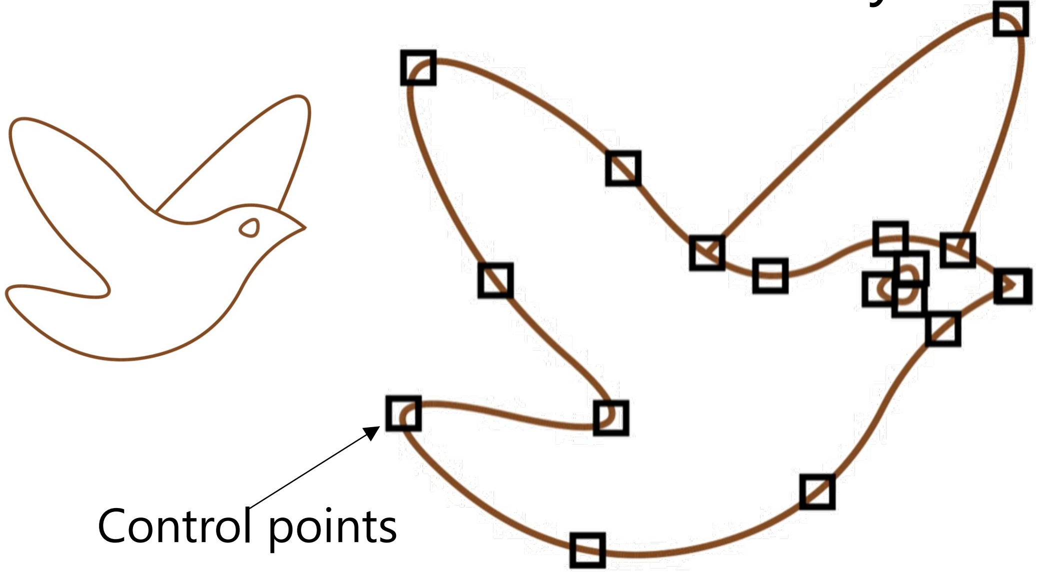
Parameterization of Catmull-Rom Curves [Yuksel,Schaefer,Keyser,CAD11]

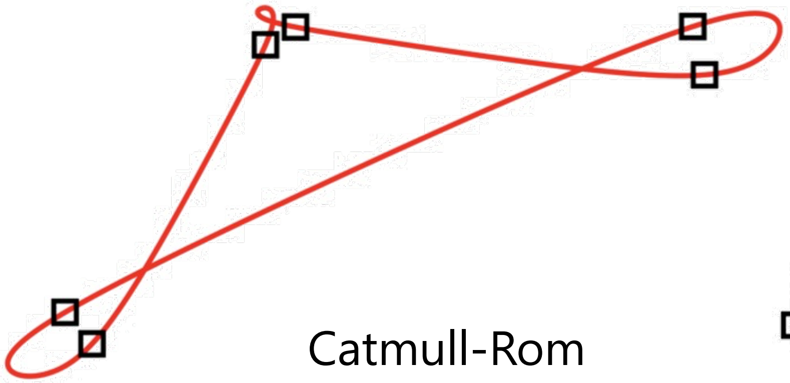
Recent paper (1): κ-Curves

- Collaboration between university & company (Adobe)
- Features:
  - C^2 continuous (smoother)
  - Curvature maxima always on control points

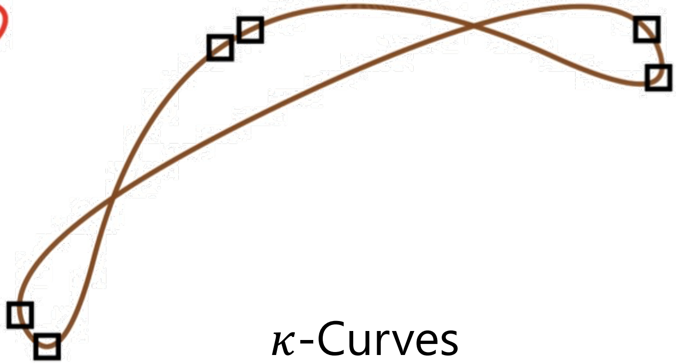


https://www.youtube.com/watch?v=MLg8xQgElfk





Catmull-Rom

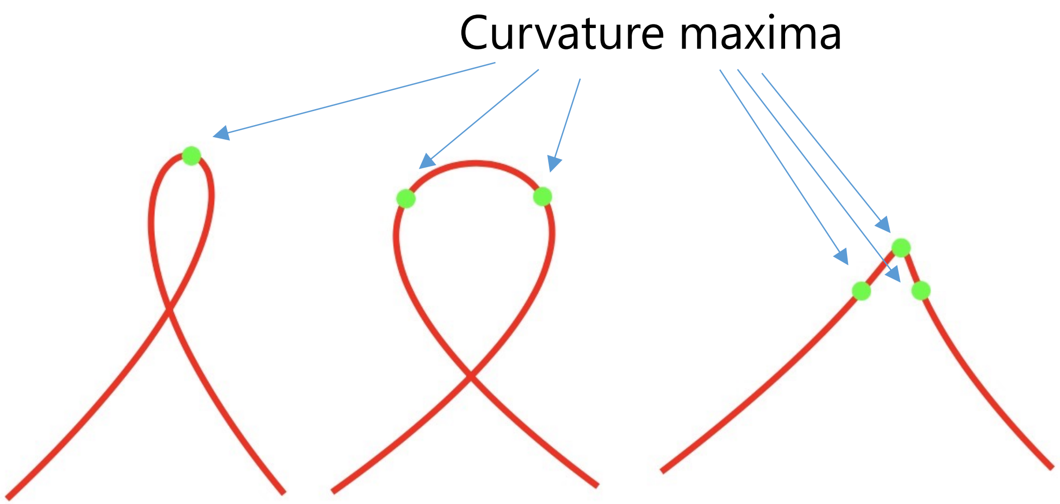


κ-Curves

κ-Curves: Interpolation at Local Maximum Curvature [Yan, Schiller, Wilensky, Carr, Schaefer, SIGGRAPH 2017]

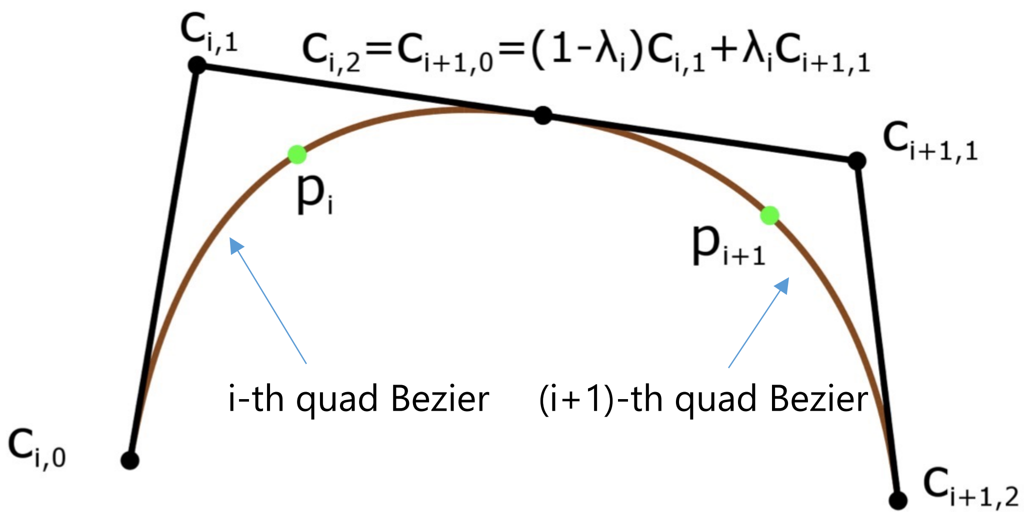
Key ideas of κ-Curves

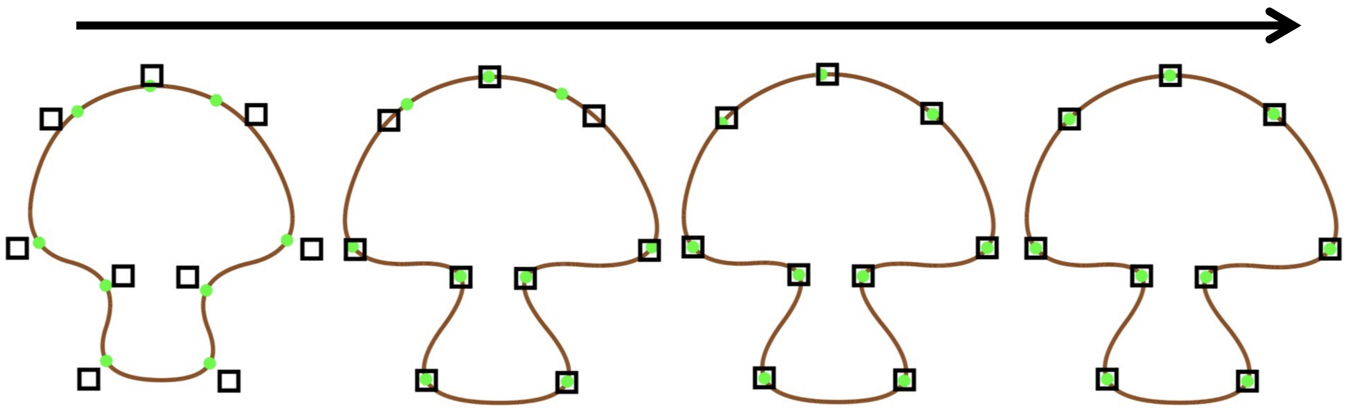
- Cubic Bezier is difficult to control



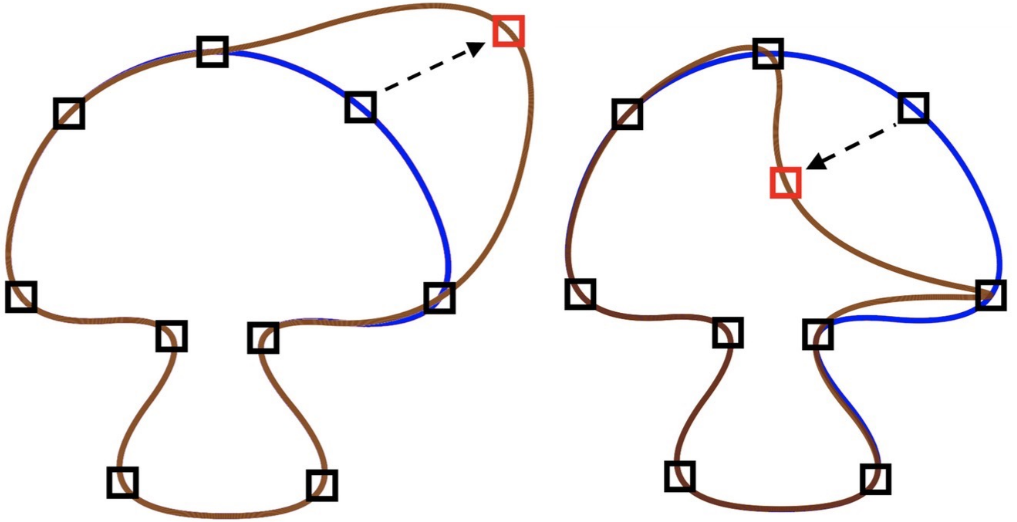
Possible configurations with cubic Bezier

- Actually, *quadratic* Bezier is easier to use!
  - At most one curvature maximum can exist
  - User specifies curvature maxima
    ➔ reverse compute control points of quadratic Bezier

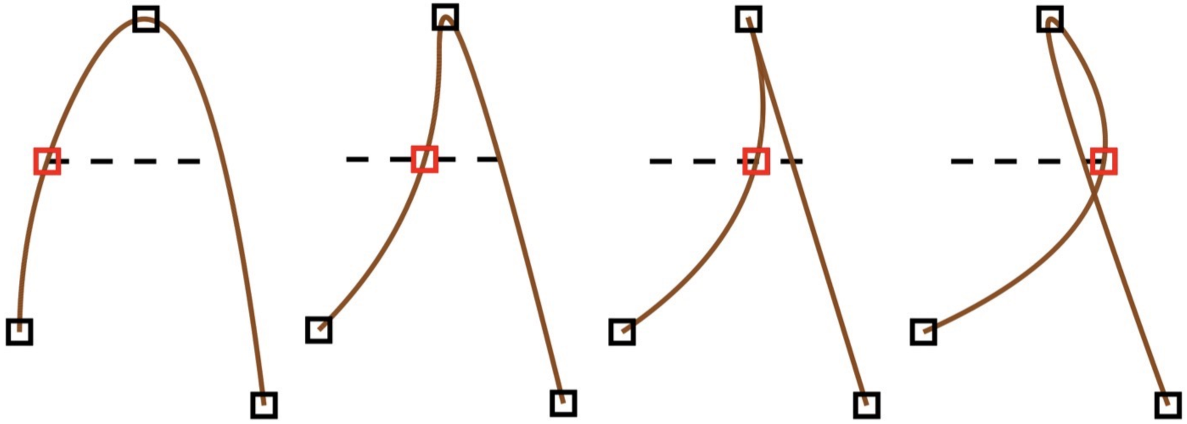




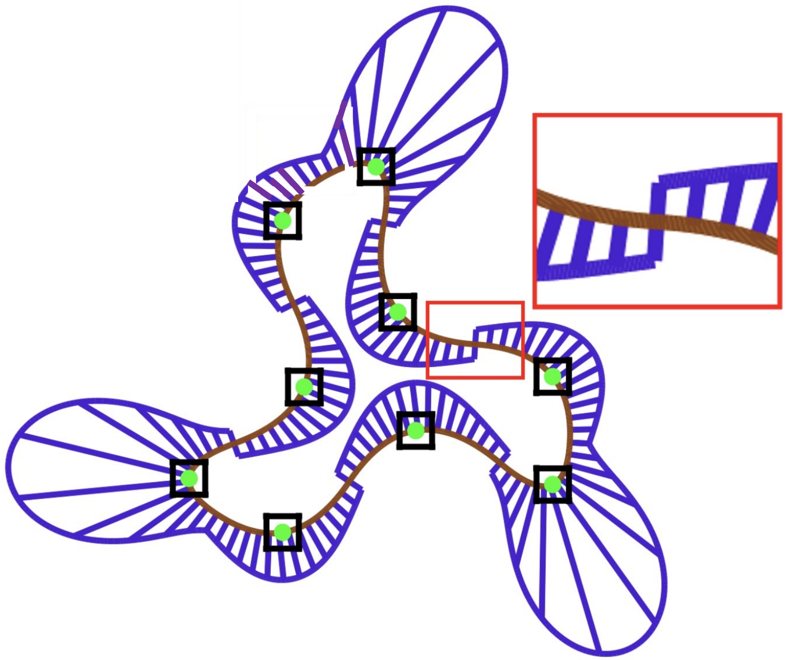
Global/nonlinear formulation = iterative computation



Change of one CP = change of entire shape

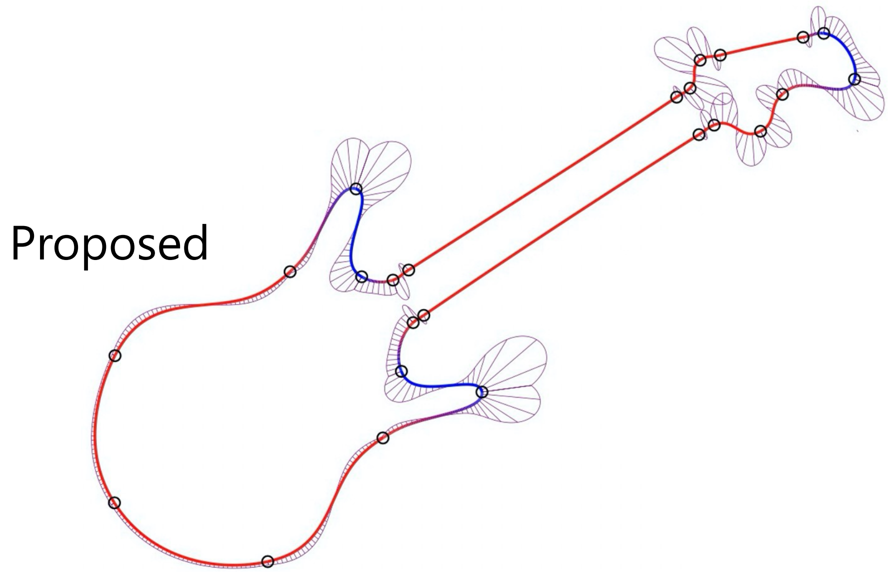


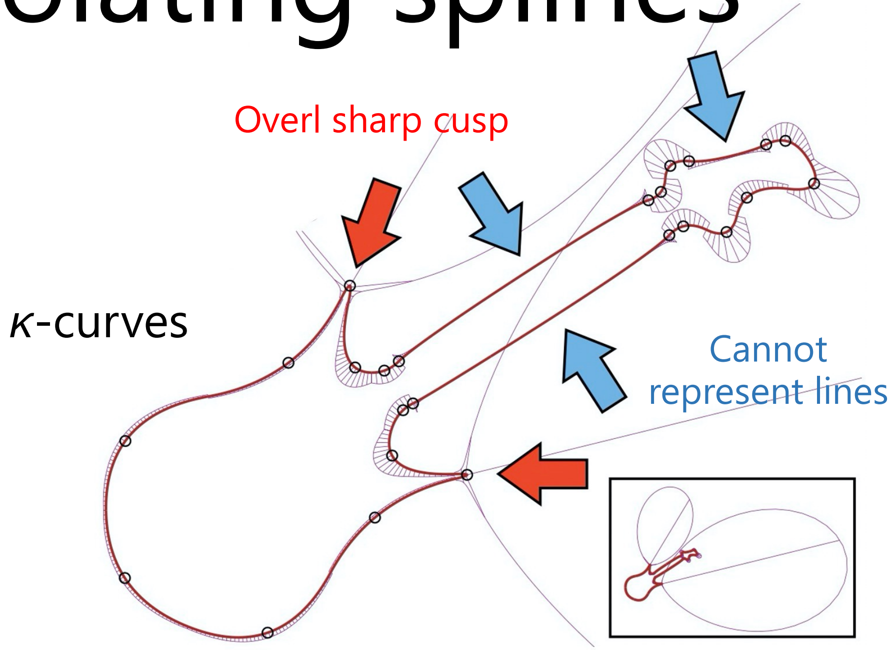
”Buckling” always occurs on CPs



Curvature discontinuity at convex/concave boundary

Recent paper (2): C^2 interpolating splines





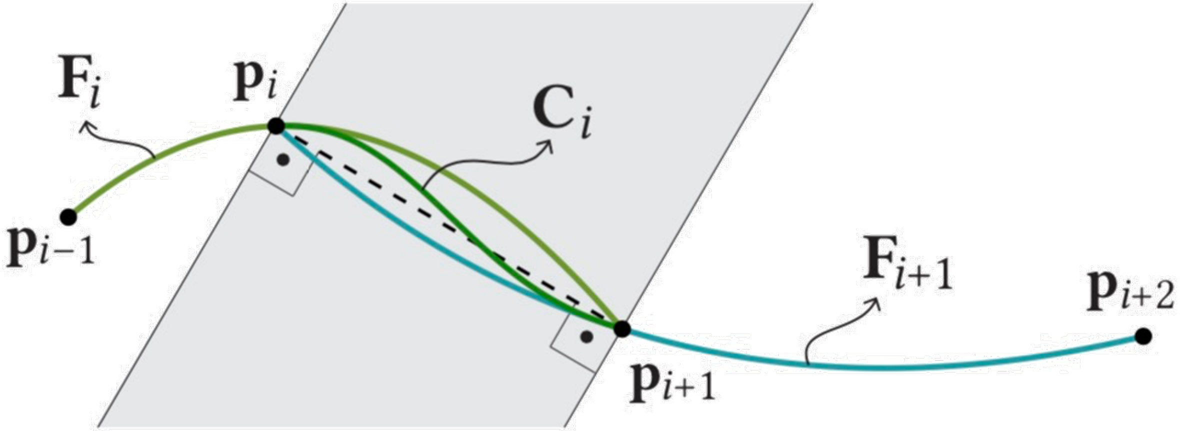
- Drawbacks of κ-curves:
  - ☹ Global optimization (high computational cost)
  - ☹ Global support (one CP moves, whole shape changes)
  - ☹ Cannot represent circular arcs or lines
  - ☹ Cannot be exnteded to 3D
- Simple method overcoming these issues

Application to hair modeling

A Class of C2 Interpolating Splines [Yuksel, SIGGRAPH 2020]

Key idea:

Define each curve segment $\mathbf{C}_i(\theta)$ by combining interpolating function $\mathbf{F}_i$ passsing through 3 CPs and trigonometric functions



$$\mathbf{F}_i(0) = \mathbf{p}_{i-1}\ , \qquad \mathbf{F}_i(\tfrac{\pi}{2}) = \mathbf{p}_i\ , \qquad \mathbf{F}_i(\pi) = \mathbf{p}_{i+1}$$

$$\mathbf{C}_i(\theta) = \cos^2\theta\, \mathbf{F}_i(\theta + \tfrac{\pi}{2}) + \sin^2\theta\, \mathbf{F}_{i+1}(\theta) \qquad 0 \le \theta \le \frac{\pi}{2}$$

➔ $\mathbf{C}_i$ determined by nearby 4 points only

1st derivative

$$\mathbf{C}_i'(\theta) = 2\cos\theta\sin\theta\ \left(\mathbf{F}_{i+1}(\theta) - \mathbf{F}_i(\theta + \tfrac{\pi}{2})\right) + \cos^2\theta\, \mathbf{F}_i'(\theta + \tfrac{\pi}{2}) + \sin^2\theta\, \mathbf{F}_{i+1}'(\theta)\ ,$$

➔ $\mathbf{C}_i'(\tfrac{\pi}{2}) = \mathbf{C}_{i+1}'(0)$

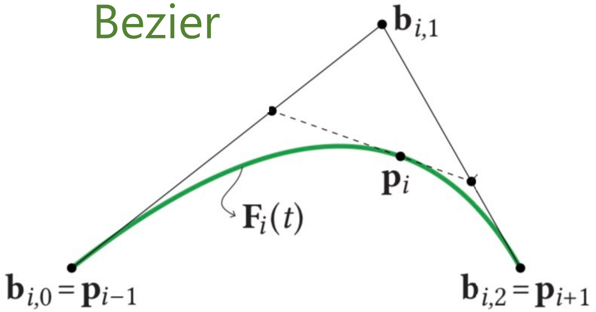
2nd derivative

$$\mathbf{C}_i''(\theta) = 2\left(\cos^2\theta - \sin^2\theta\right)\ \left(\mathbf{F}_{i+1}(\theta) - \mathbf{F}_i(\theta + \tfrac{\pi}{2})\right) + 4\cos\theta\sin\theta\ \left(\mathbf{F}_{i+1}'(\theta) - \mathbf{F}_i'(\theta + \tfrac{\pi}{2})\right) + \cos^2\theta\, \mathbf{F}_i''(\theta + \tfrac{\pi}{2}) + \sin^2\theta\, \mathbf{F}_{i+1}''(\theta)\ .$$

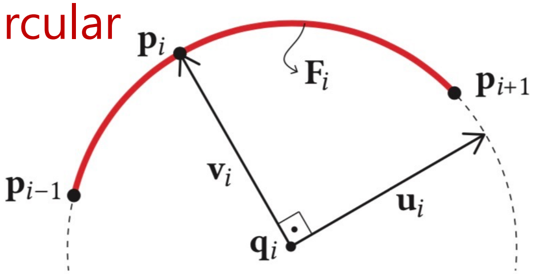
➔ $\mathbf{C}_i''(\tfrac{\pi}{2}) = \mathbf{C}_{i+1}''(0)$

3 types of interpolating functions

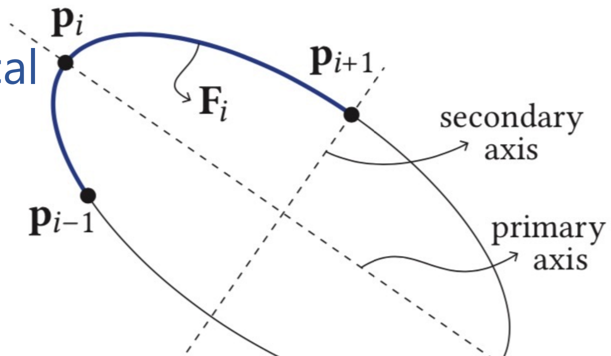
Bezier



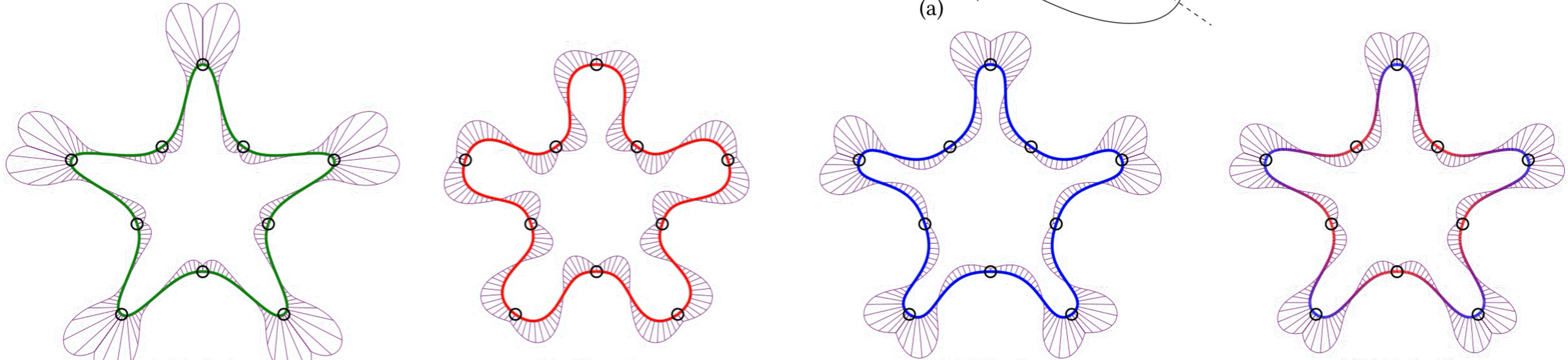
Circular



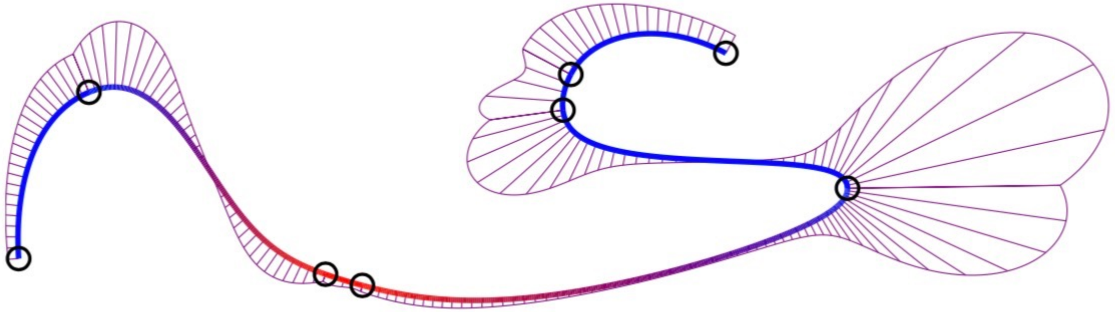
Elliptical



(a)



(a) Bézier (b) Circular (c) Elliptical (d) Hybrid



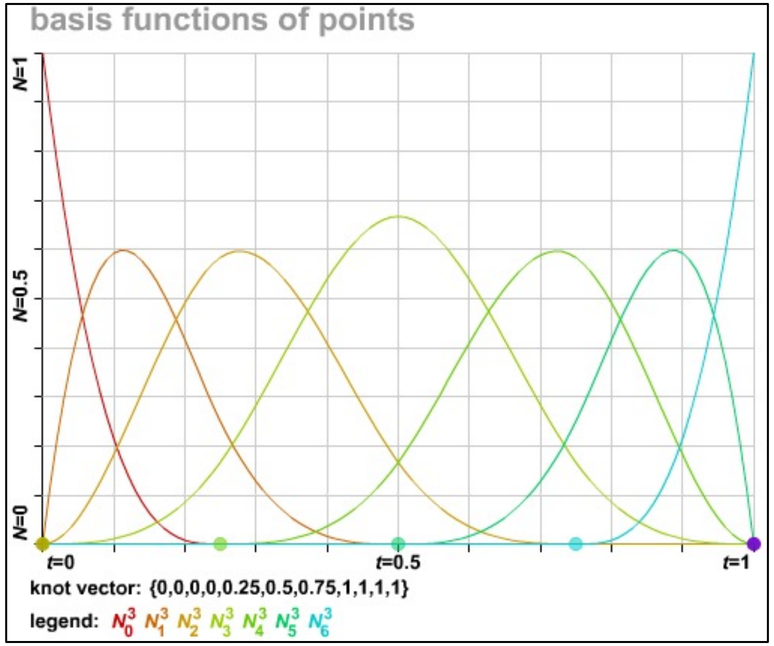
Hybrid:

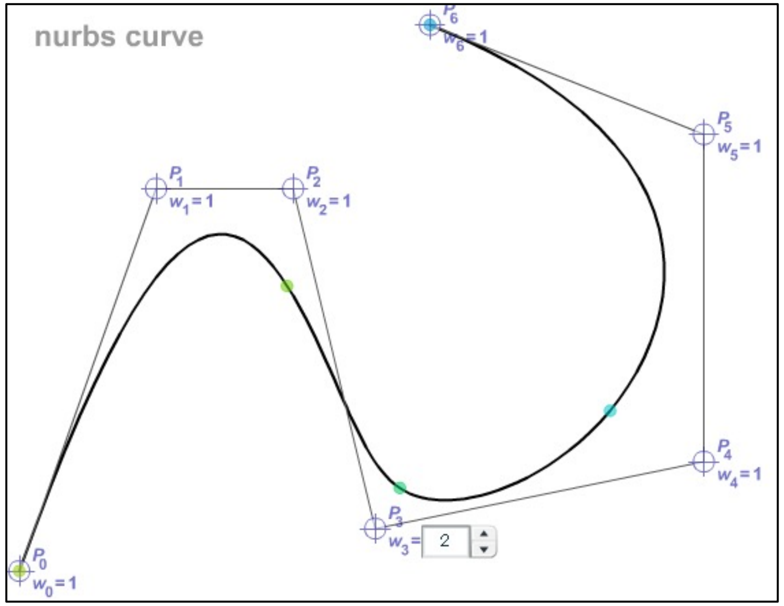
Highly curved part ➔ Elliptic

Mostly flat part ➔ Circular

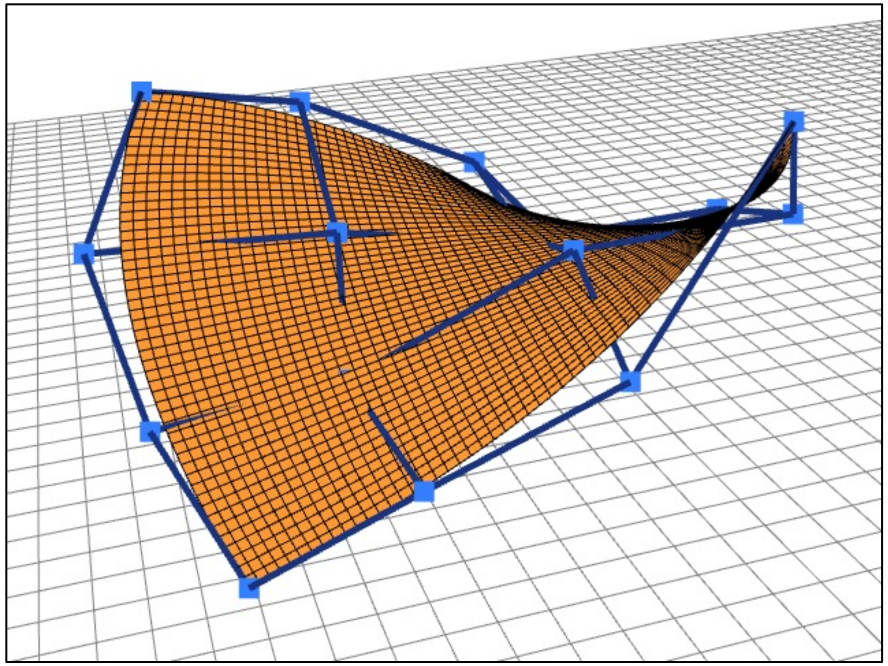
B-spline

- Another way of defining polynomial spline
  - Represent curve as sum of **basis functions**
  - Cubic basis is the most commonly used
  - Deeply related to subdivision surfaces
    ➔ Next lecture
- **N**on-**U**niform **R**ational **B-S**pline
  - Non-Uniform = varying spacing of knots (t_k)
  - Rational = arbitrary weights for CPs
  - (Complex stuff, not covered)
- Cool Flash demo: http://geometrie.foretnik.net/files/NURBS-en.swf
  - SWF player: https://ruffle.rs/demo/





Parametric surfaces



- One parameter ➔ Curve $P(t)$
- Two parameters ➔ Surface $P(s,t)$

- Cubic Bezier surface:
  - Input: 4×4=16 control points P_{ij}

$$P(s,t) = \sum_{i=0}^{3}\sum_{j=0}^{3} b_i^3(s) b_j^3(t) P_{ij}$$

Bernstein basis functions

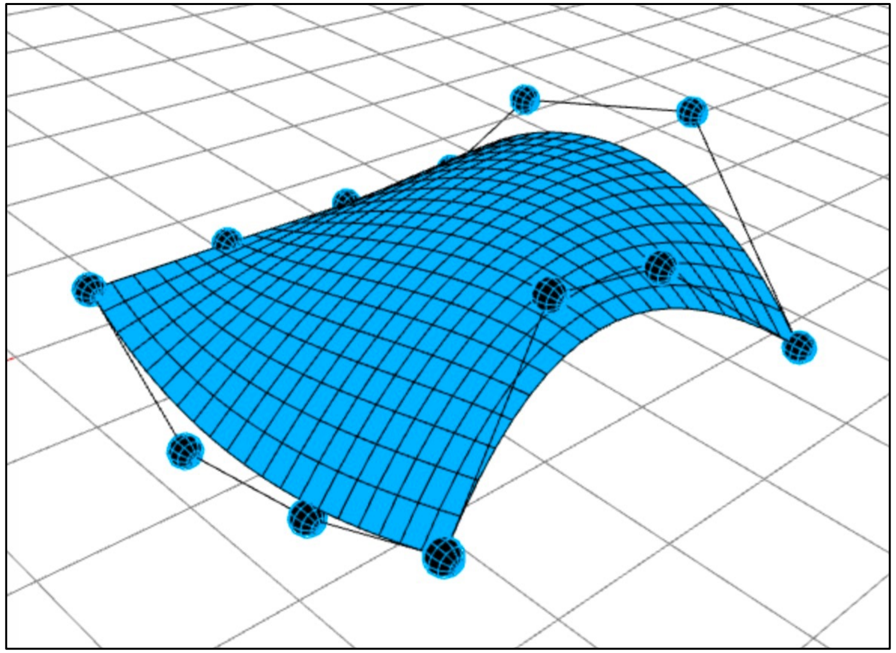
$b_0^3(t) = (1-t)^3$

$b_1^3(t) = 3t(1-t)^2$

$b_2^3(t) = 3t^2(1-t)$

$b_3^3(t) = t^3$

Coons patch

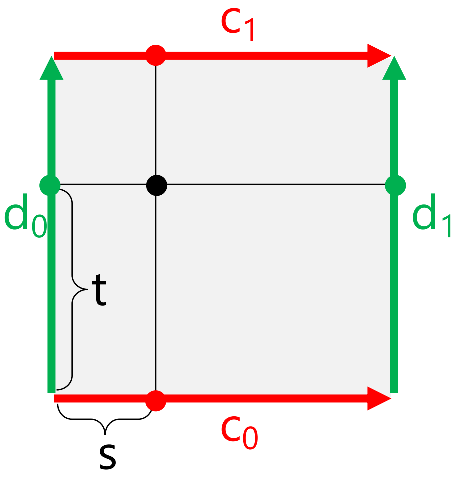


- Given four curves joining at endpoints, evaluate pairs of opposing curves, then interpolate them ➔ a surface patch

$$L_c(s,t) = (1-t)c_0(s) + tc_1(s)$$

$$L_d(s,t) = (1-s)d_0(t) + sd_1(t)$$

Patch function: $\boxed{C(s,t) = L_c(s,t) + L_d(s,t) - B(s,t)}$



where

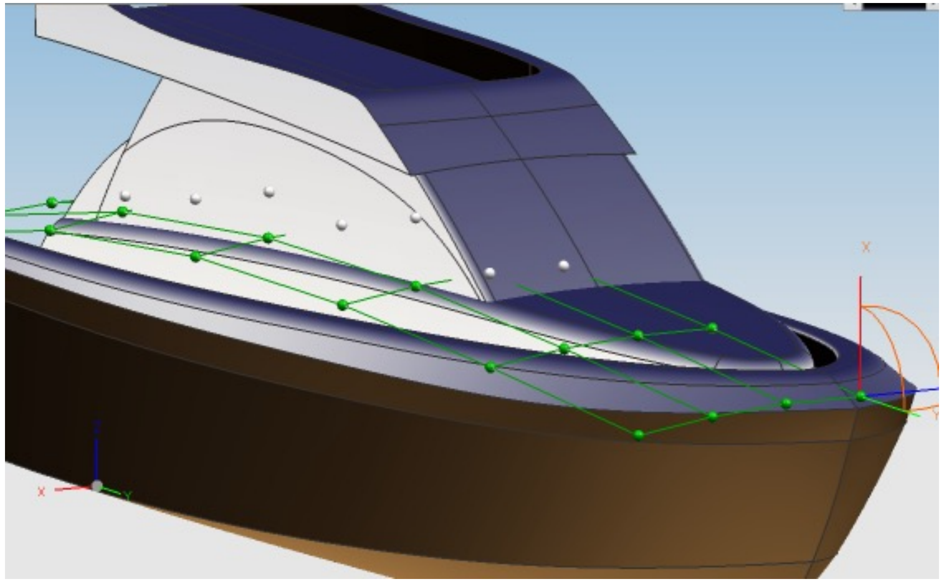
$$B(s,t) = c_0(0)(1-s)(1-t) + c_0(1)s(1-t) + c_1(0)(1-s)t + c_1(1)st$$

is the **bilinear interpolation** of four corners

https://en.wikipedia.org/wiki/Coons_patch

3D modeling using parametric surface patches

- Pros
  - Can compactly represent smooth surfaces
  - Can accurately represent spheres, cones, etc



- Cons
  - Hard to design nice layout of patches
  - Hard to maintain continuity across patches
- Often used for designing man-made objects consisting of simple parts

Pointers

- http://en.wikipedia.org/wiki/Bezier_curve
- http://agg.sourceforge.net/antigrain.com/research/adaptive_bezier/index.html
- http://en.wikipedia.org/wiki/Cubic_Hermite_spline
- http://en.wikipedia.org/wiki/Centripetal_Catmull%E2%80%93Rom_spline