On quaternions

Rotation about arbitrary axis

- Needed in various situations (e.g. camera manipulation)

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

about X-axis

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

about Y-axis

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

about Z-axis

$$R = \begin{bmatrix} \cos\theta + u_x^2(1-\cos\theta) & u_x u_y(1-\cos\theta) - u_z\sin\theta & u_x u_z(1-\cos\theta) + u_y\sin\theta \\ u_y u_x(1-\cos\theta) + u_z\sin\theta & \cos\theta + u_y^2(1-\cos\theta) & u_y u_z(1-\cos\theta) - u_x\sin\theta \\ u_z u_x(1-\cos\theta) - u_y\sin\theta & u_z u_y(1-\cos\theta) + u_x\sin\theta & \cos\theta + u_z^2(1-\cos\theta) \end{bmatrix}.$$

about arbitrary axis

(u_x, u_y, u_z) : axis vector

- Problems with matrix representation
  - Overly complex!
    - Should be represented by 2 DoF (axis direction) + 1 DoF (angle) = 3 DoF
  - Can't handle interpolation (blending) well

Degree **o**f **F**reedom

Complex number & quaternion

- Complex number
  - $\mathbf{i}^2 = -1$
  - $\mathbf{c} = (a, b) \coloneqq a + b\,\mathbf{i}$
  - $\mathbf{c}_1\mathbf{c}_2 = (a_1, b_1)(a_2, b_2) = a_1a_2 - b_1b_2 + (a_1b_2 + b_1a_2)\,\mathbf{i}$

- Quaternion
  - $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$
    - $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$ Not commutative!
  - $\mathbf{q} = (a, b, c, d) \coloneqq a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$
  - $\mathbf{q}_1\mathbf{q}_2 = (a_1, b_1, c_1, d_1)(a_2, b_2, c_2, d_2)$

$$= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\,\mathbf{i}$$

$$+ (a_1c_2 + c_1a_2 + d_1b_2 - b_1d_2)\,\mathbf{j} + (a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)\,\mathbf{k}$$

Expressed as pair of scalar & vector

$s \in \mathbb{R}$

$\vec{v} \coloneqq (v_\mathrm{x}, v_\mathrm{y}, v_\mathrm{z}) \in \mathbb{R}^3$

- $\mathbf{q} = s + v_\mathrm{x}\,\mathbf{i} + v_\mathrm{y}\,\mathbf{j} + v_\mathrm{z}\,\mathbf{k} =: s + \vec{v} =: (\underbrace{s}_{\text{Scalar part}}, \underbrace{\vec{v}}_{\text{Vector part}}) \quad \in \mathbb{H}$

Scalar part

Vector part

- $\mathbf{q}_1\mathbf{q}_2 = (s_1,\ \overrightarrow{v_1})(s_2,\ \overrightarrow{v_2})$

$$= (s_1 s_2 - \overrightarrow{v_1} \cdot \overrightarrow{v_2},\ s_1\overrightarrow{v_2} + s_2\overrightarrow{v_1} + \underbrace{\overrightarrow{v_1} \times \overrightarrow{v_2}})$$

Cross-product makes it non-commutative

Conjugate, norm, inverse

- Complex number $\mathbf{c} \coloneqq (a, b) \in \mathbb{C}$
  - $\bar{\mathbf{c}} \coloneqq (a, -b)$
  - $\mathbf{c}\bar{\mathbf{c}} = (a, b)(a, -b) = (a^2 + b^2, 0) =: |\mathbf{c}|^2$
  - $\mathbf{c}\underbrace{(\bar{\mathbf{c}}/|\mathbf{c}|^2)}_{\substack{\shortparallel \\ \mathbf{c}^{-1}}} = 1$

- Quaternion $\mathbf{q} \coloneqq (s, \vec{v}) \in \mathbb{H}$
  - $\overline{\mathbf{q}} \coloneqq (s, -\vec{v})$
  - $\mathbf{q}\overline{\mathbf{q}} = (s, \vec{v})(s, -\vec{v}) = (s^2 + |\vec{v}|^2, 0) =: |\mathbf{q}|^2$
  - $\mathbf{q}\underbrace{(\overline{\mathbf{q}}/|\mathbf{q}|^2)}_{\substack{\shortparallel \\ \mathbf{q}^{-1}}} = 1$

In particular, if $|\mathbf{q}| = 1$ then $\mathbf{q}^{-1} = \overline{\mathbf{q}}$

Quaternion representing rotation about axis $\vec{u}$

- $\mathbf{q} := (\cos\theta\,,\ \vec{u}\sin\theta)$ where $|\vec{u}| = 1$, i.e. $|\mathbf{q}| = 1$

- Why???

- Consider two planes in the space of quaternions $\mathbb{H}$:
  - $P_{\parallel} := \{\, (\alpha,\ \beta\vec{u}) \mid \alpha, \beta \in \mathbb{R} \,\} \subset \mathbb{H}$
  - $P_{\perp} := \left\{ \left(0,\ \alpha\overrightarrow{u_\perp} + \beta(\vec{u}\times\overrightarrow{u_\perp})\right) \mid \alpha, \beta \in \mathbb{R} \right\} \subset \mathbb{H}$ — $\overrightarrow{u_\perp}$: arbitrary unit vector orthogonal to $\vec{u}$

- How does $\mathbf{q}$ affect quaternions belonging to these planes?

Original explanation by Dr. Scott Schaefer https://people.engr.tamu.edu/schaefer/teaching/441_Fall2017/lectures/Quaternions.ppt

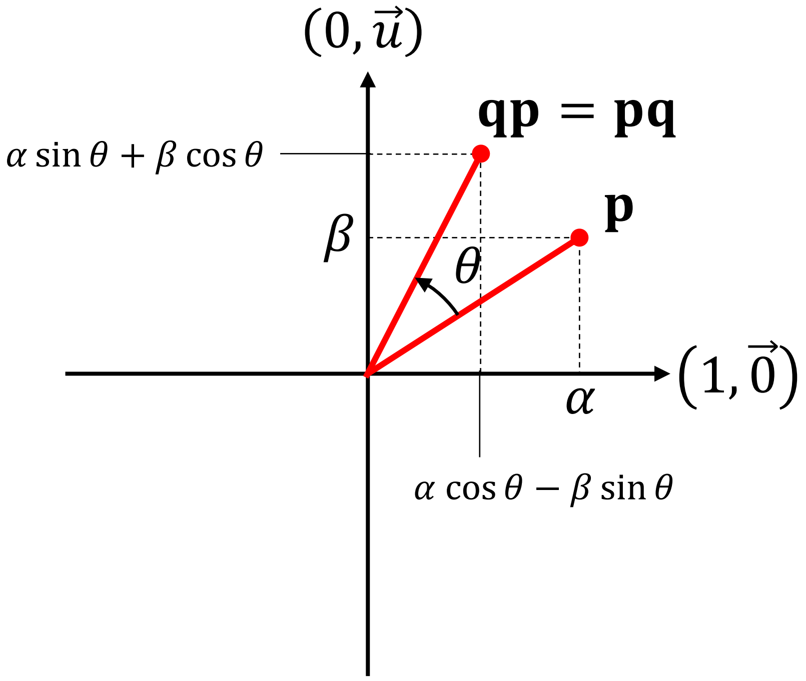
Multiplying $\mathbf{q}$ to $\mathbf{p} \in P_{\parallel}$

- $\mathbf{q} \coloneqq (\cos\theta\,,\ \vec{u}\sin\theta)$
- $\mathbf{p} \coloneqq (\alpha,\ \beta\vec{u}) \in P_{\parallel}$

- From left:
  - $\mathbf{qp} = (\cos\theta\,,\ \vec{u}\sin\theta)(\alpha,\ \beta\vec{u})$
    $= (\alpha\cos\theta - \beta\sin\theta\,, (\alpha\sin\theta + \beta\cos\theta)\vec{u})$

- From right:
  - $\mathbf{pq} = (\alpha,\ \beta\vec{u})(\cos\theta\,,\ \vec{u}\sin\theta)$
    $= (\alpha\cos\theta - \beta\sin\theta\,, (\alpha\sin\theta + \beta\cos\theta)\vec{u})$
    $= \mathbf{qp}$

- $\mathbf{qp\bar{q}} = \mathbf{\bar{q}}(\mathbf{qp}) = \mathbf{p}$



Multiplying $\mathbf{q}$ to $\mathbf{p} \in P_\perp$

- $\mathbf{q} \coloneqq (\cos\theta\,,\ \vec{u}\sin\theta)$
- $\mathbf{p} \coloneqq \left(0,\ \alpha\overrightarrow{u_\perp} + \beta(\vec{u}\times\overrightarrow{u_\perp})\right) \in P_\perp$

- From left:
  - $\mathbf{qp} = (\cos\theta\,,\ \vec{u}\sin\theta)\left(0,\ \alpha\overrightarrow{u_\perp} + \beta(\vec{u}\times\overrightarrow{u_\perp})\right)$

$$= \left(0, \cos\theta\left(\alpha\overrightarrow{u_\perp} + \beta(\vec{u}\times\overrightarrow{u_\perp})\right) + (\vec{u}\sin\theta)\times\left(\alpha\overrightarrow{u_\perp} + \beta(\vec{u}\times\overrightarrow{u_\perp})\right)\right)$$

$$= \left(0, (\alpha\cos\theta - \beta\sin\theta)\overrightarrow{u_\perp} + (\alpha\sin\theta + \beta\cos\theta)(\vec{u}\times\overrightarrow{u_\perp})\right)$$

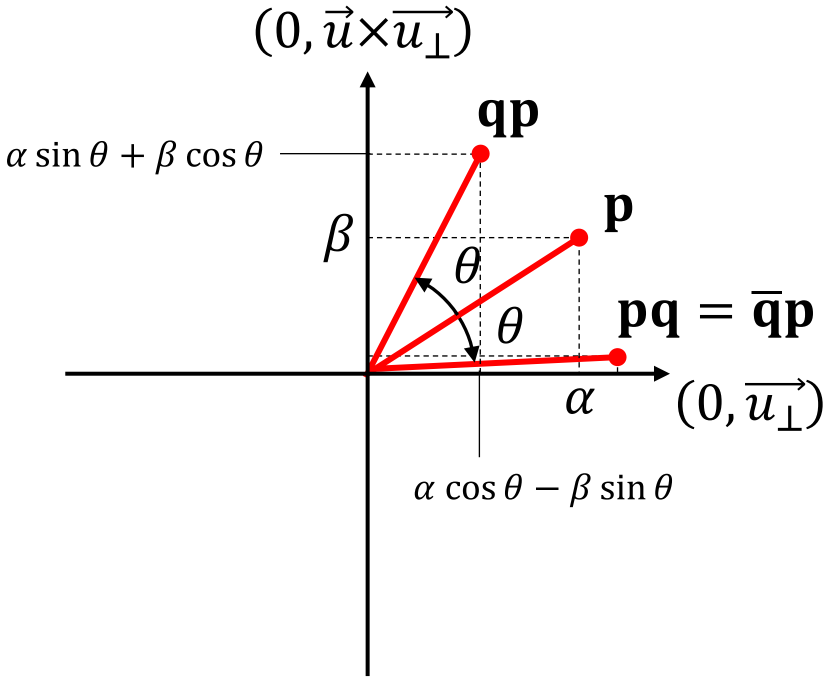
$\vec{u}\times(\vec{u}\times\overrightarrow{u_\perp}) = -\overrightarrow{u_\perp}$

- From right:
  - $\mathbf{pq} = \left(0,\ \alpha\overrightarrow{u_\perp} + \beta(\vec{u}\times\overrightarrow{u_\perp})\right)(\cos\theta\,,\ \vec{u}\sin\theta)$

$$= \left(0, (\alpha\cos\theta + \beta\sin\theta)\overrightarrow{u_\perp} + (-\alpha\sin\theta + \beta\cos\theta)(\vec{u}\times\overrightarrow{u_\perp})\right)$$

$$= \bar{\mathbf{q}}\mathbf{p}$$

- $\mathbf{qp}\bar{\mathbf{q}} = \overline{(\bar{\mathbf{q}})}(\mathbf{qp}) = \mathbf{q}^2\mathbf{p}$



Rotating arbitrary 3D position $\vec{p} \in \mathbb{R}^3$ by $\mathbf{q}$

- Can always decompose $\vec{p}$ into linear combination of $\vec{u}$, $\overrightarrow{u_\perp}$, $\vec{u}\times\overrightarrow{u_\perp}$:
  - $\vec{p} = \alpha\overrightarrow{u_\perp} + \beta(\vec{u}\times\overrightarrow{u_\perp}) + \gamma\vec{u}$

- $$\mathbf{p} \coloneqq (0, \vec{p}) = \underbrace{(0, \gamma\vec{u})}_{\mathbf{p}_\parallel \in P_\parallel} + \underbrace{\left(0, \alpha\overrightarrow{u_\perp} + \beta(\vec{u}\times\overrightarrow{u_\perp})\right)}_{\mathbf{p}_\perp \in P_\perp}$$

- $$\begin{aligned} \mathbf{q}\mathbf{p}\overline{\mathbf{q}} &= \mathbf{q}(\mathbf{p}_\parallel + \mathbf{p}_\perp)\overline{\mathbf{q}} = \mathbf{q}\mathbf{p}_\parallel\overline{\mathbf{q}} + \mathbf{q}\mathbf{p}_\perp\overline{\mathbf{q}} \\ &= \mathbf{p}_\parallel + \mathbf{q}^2\mathbf{p}_\perp \\ &= (0, \underbrace{(\alpha\cos 2\theta - \beta\sin 2\theta)\overrightarrow{u_\perp} + (\alpha\sin 2\theta + \beta\cos 2\theta)(\vec{u}\times\overrightarrow{u_\perp}) + \gamma\vec{u})}_{\text{Result of rotating } \vec{p} \text{ about } \vec{u} \text{ by } 2\theta} \end{aligned}$$

- To make it rotate by θ, use $\mathbf{q} \coloneqq \left(\cos\frac{\theta}{2},\ \vec{u}\sin\frac{\theta}{2}\right)$

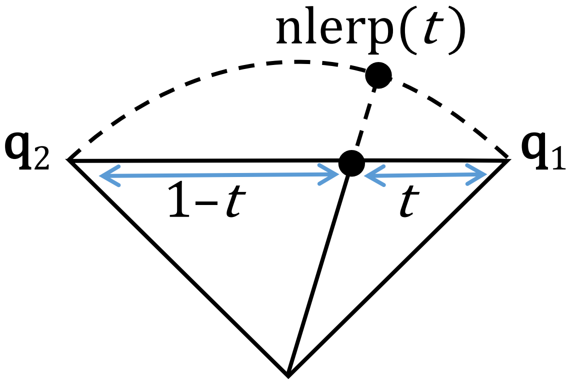
Matrix vs quaternion

	Matrix	Quaternion
Size	9	4
# of multiplications needed for perfoming rotation of 3D position	9	28
# of multiplications needed for compositing rotations	27	16

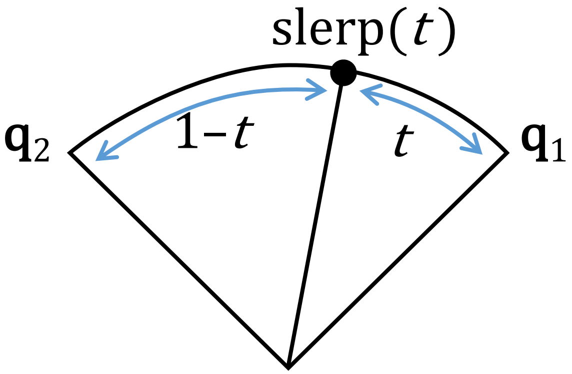
- Use quaternions for compositing or interpolating rotations
- Use matrix for final coordinate calculation

Rotation interpolation using quaternions

- Linear interp + normalization (nlerp)
  - $\text{nlerp}(\mathbf{q}_1, \mathbf{q}_2, t) \coloneqq \text{normalize}\big((1-t)\mathbf{q}_1 + t\,\mathbf{q}_2\big)$
  - ☺less computation, ☹non-uniform angular speed

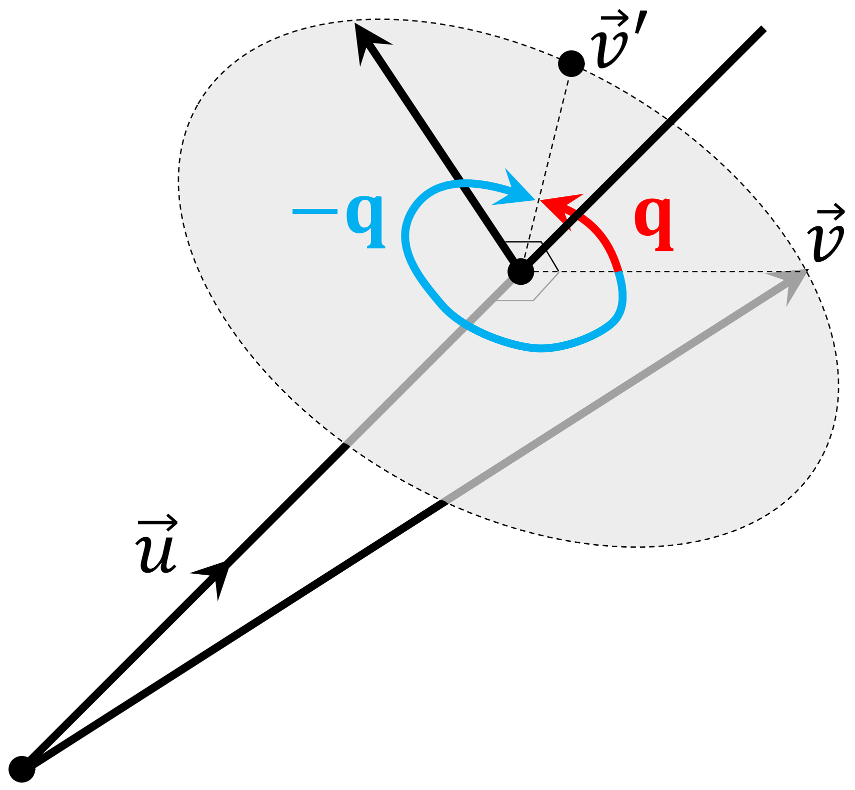


- Spherical linear interpolation (slerp)
  - $\Omega = \cos^{-1}(\mathbf{q}_1 \cdot \mathbf{q}_2)$
  - $\text{slerp}(\mathbf{q}_1, \mathbf{q}_2, t) \coloneqq \frac{\sin(1-t)\Omega}{\sin\Omega}\mathbf{q}_1 + \frac{\sin t\Omega}{\sin\Omega}\mathbf{q}_2$
  - ☹more computation, ☺constant angular speed



Animating rotation with quaternion curves. Shoemake, SIGGRAPH 1985
http://number-none.com/product/Understanding%20Slerp,%20Then%20Not%20Using%20It/

Signs of quaternions



- Quaternion with angle θ:
  - $\mathbf{q} = \cos\frac{\theta}{2} + \vec{u}\sin\frac{\theta}{2}$
- Quaternion with angle $\theta - 2\pi$:
  - $\cos\frac{\theta-2\pi}{2} + \vec{u}\sin\frac{\theta-2\pi}{2} = -\mathbf{q}$
- When interpolating from $\mathbf{q}_1$ to $\mathbf{q}_2$, negate $\mathbf{q}_2$ if $\mathbf{q}_1 \cdot \mathbf{q}_2$ is negative
  - Otherwise, the interpolation path becomes longer