Mathematics and Implementation of
Computer Graphics Techniques 2015

Boundary Aligned Smooth 3D Cross-Frame Field

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SIGGRAPH Asia 2011

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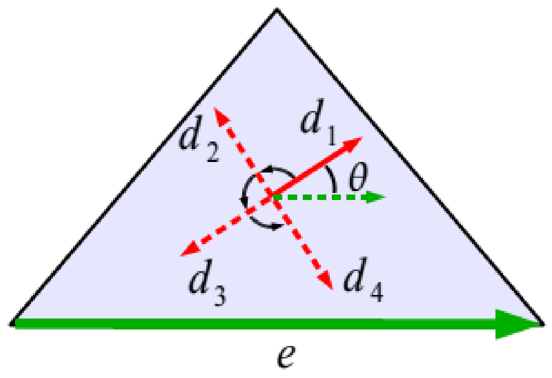
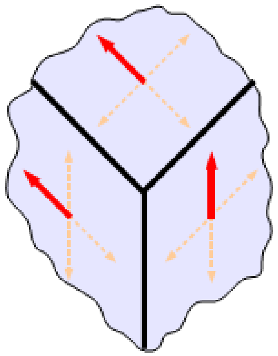
2nd round

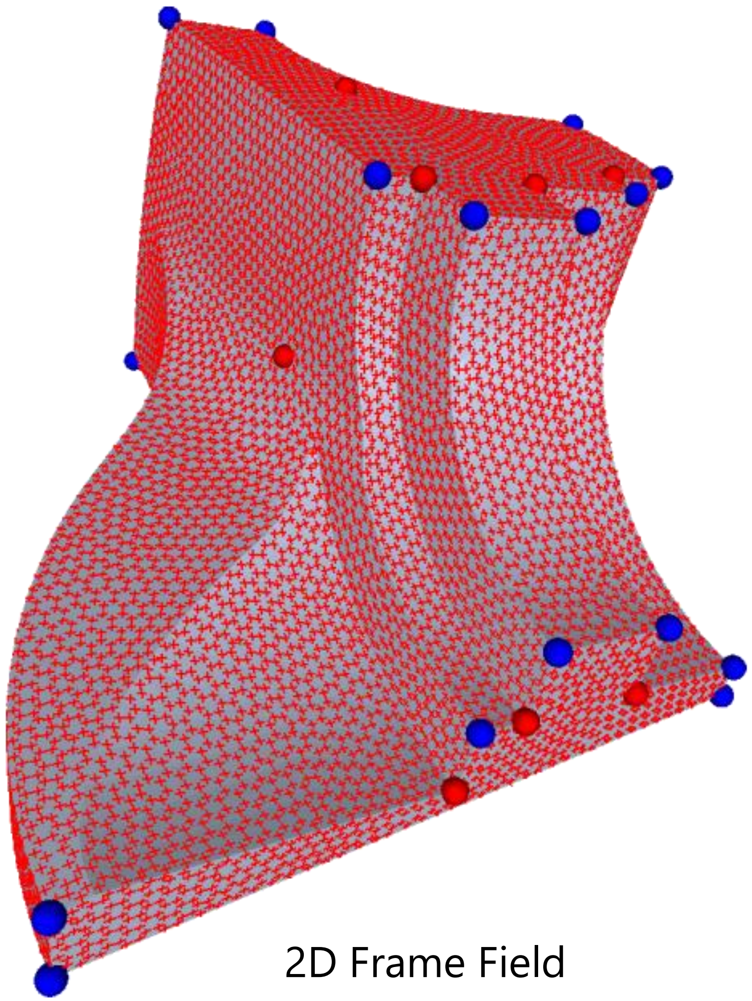
3 October 2015

Background: 2D Frame Field & Quad Meshing

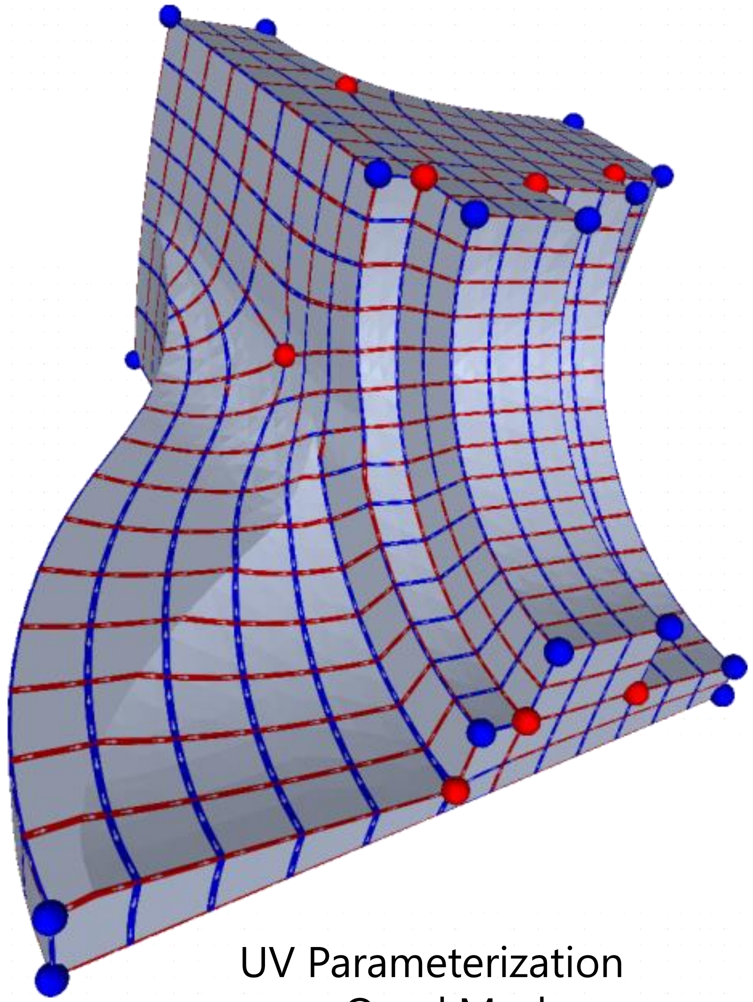




- 2D Frame Field
  - 🡐 Auto-computed
- UV Parameterization
  - 🡐 Auto-computed



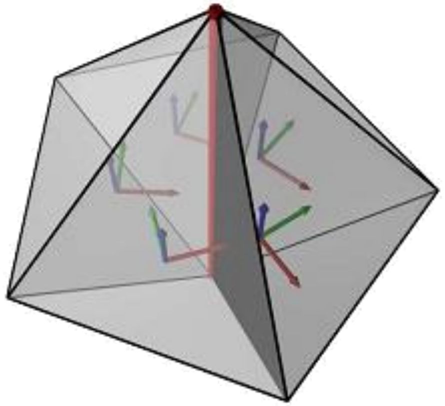
2D Frame Field

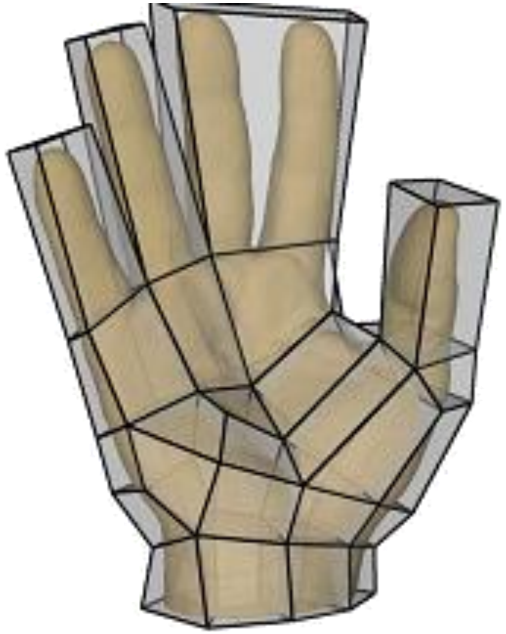


UV Parameterization
= Quad Mesh

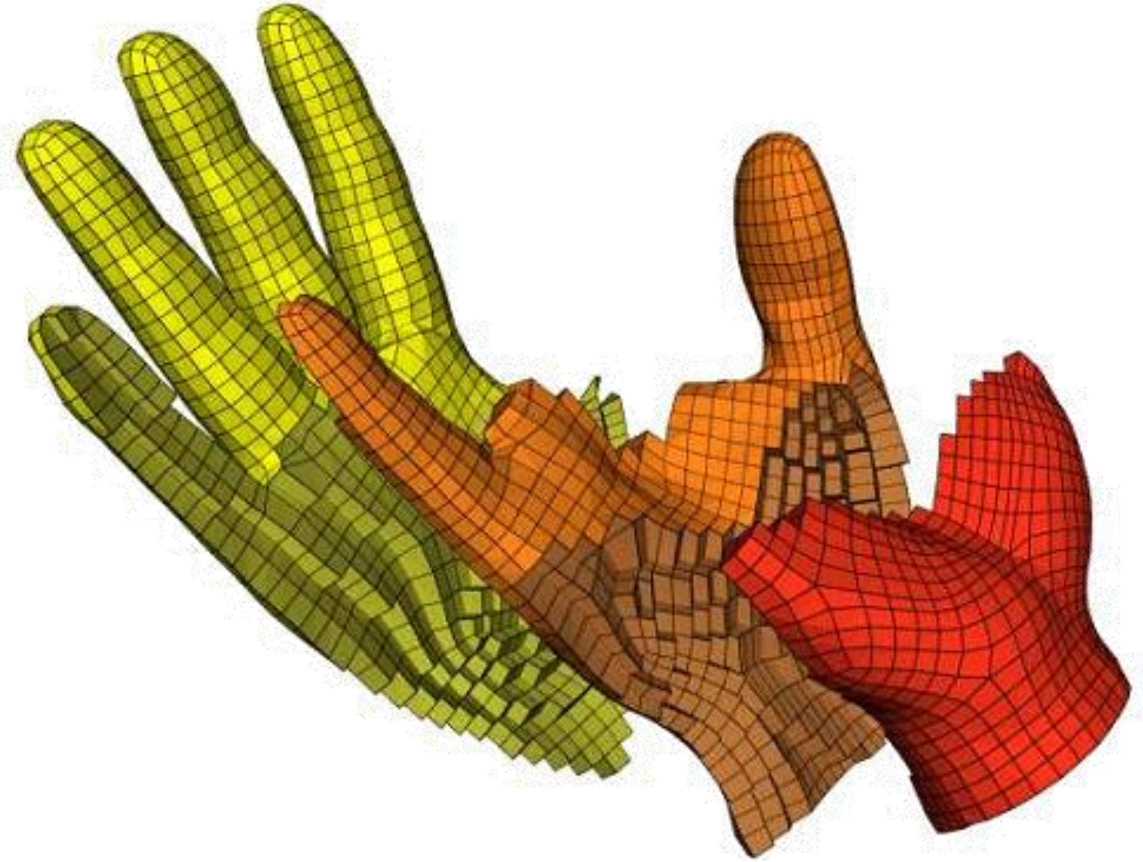
Mixed-Integer Quadrangulation [Bommes,Zimmer,Kobbelt,SIGGRAPH09]

Background: 3D Frame Field & Hex Meshing





"Meta-Mesh" to define
3D Frame Field

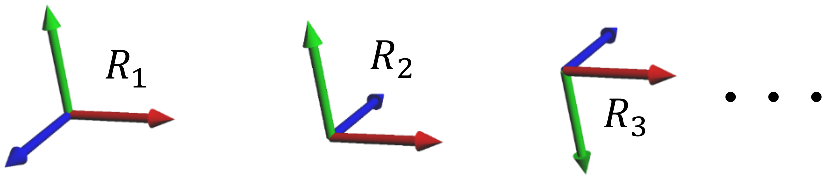


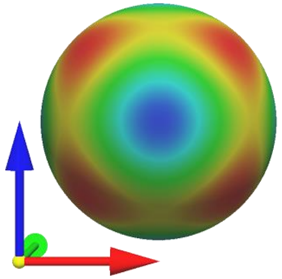
UVW Parameterization
= Hex Mesh

- 3D Frame Field
  - 🡐 Heuristic
- UVW Parameterization
  - 🡐 Auto-computed

CubeCover - Parameterization of 3D Volumes [Nieser,Reitebuch,Polthier,SGP11]

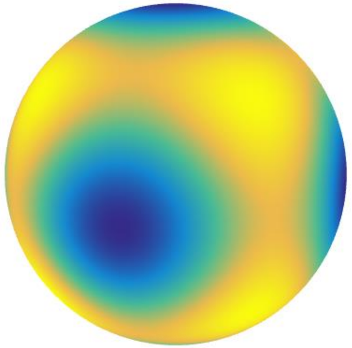
Distance between 3D Frames



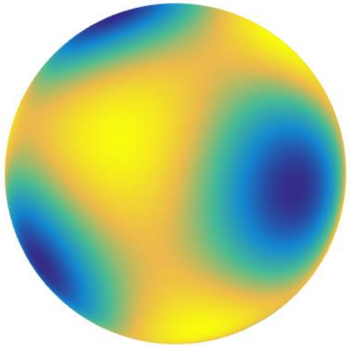


$$h(s) := s_x^2 s_y^2 + s_y^2 s_z^2 + s_z^2 s_x^2$$

$$d(R_a, R_b) := \int_{s \in S^2} \left(h(R_a s) - h(R_b s) \right)^2 ds$$



$h(R_a s)$

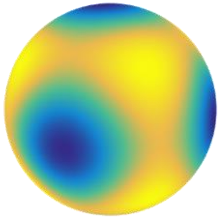


$h(R_b s)$

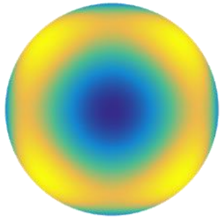
- Integral over an entire sphere ➔ **S**pherical **H**armonics!

Representing 3D frames using SH coeffs

$$h(R\,s) = \lambda_{-4} \times Y_4^{-4} + \lambda_{-3} \times Y_4^{-3} + \lambda_{-2} \times Y_4^{-2} + \lambda_{-1} \times Y_4^{-1} + \lambda_0 \times Y_4^{0} + \lambda_1 \times Y_4^{1} + \lambda_2 \times Y_4^{2} + \lambda_3 \times Y_4^{3} + \lambda_4 \times Y_4^{4}$$



$$h(s) = s_x^2 s_y^2 + s_y^2 s_z^2 + s_z^2 s_x^2$$



$$= \sqrt{7} \times Y_4^0 + \sqrt{5} \times Y_4^4$$

$$\begin{pmatrix} \lambda_{-4} \\ \lambda_{-3} \\ \lambda_{-2} \\ \lambda_{-1} \\ \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} \\ \\ & \hat{R} & \\ \\ \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{7} \\ 0 \\ 0 \\ 0 \\ \sqrt{5} \end{pmatrix}$$

Representing $\hat{R}$ using ZYZ Euler angles

$$R(\alpha,\beta,\gamma) = R_Z(\gamma)\, R_Y(\beta)\, R_Z(\alpha)$$

Rotation about Z axis by α

$$= R_Z(\gamma)\, R_X\left(-\frac{\pi}{2}\right) R_Z(\beta)\, R_X\left(\frac{\pi}{2}\right) R_Z(\alpha)$$

$$= \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{R}(\alpha,\beta,\gamma) = \hat{R}_Z(\gamma)\, \hat{R}_X\left(-\frac{\pi}{2}\right) \hat{R}_Z(\beta)\, \hat{R}_X\left(\frac{\pi}{2}\right) \hat{R}_Z(\alpha)$$

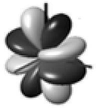
Deriving $\hat{R}_{\mathrm{Z}}(\alpha)$ & $\hat{R}_{\mathrm{X}}\left(\frac{\pi}{2}\right)$

https://en.wikipedia.org/wiki/Table_of_spherical_harmonics#Real_spherical_harmonics


$Y_4^{-4}(x,y,z) = \frac{3}{4}\sqrt{\frac{35}{\pi}}\,xy(x^2-y^2)$

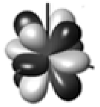

$Y_4^{-3}(x,y,z) = \frac{3}{4}\sqrt{\frac{35}{2\pi}}\,(3x^2-y^2)yz$


$Y_4^{-2}(x,y,z) = \frac{3}{4}\sqrt{\frac{5}{\pi}}\,xy(7z^2-1)$


$Y_4^{-1}(x,y,z) = \frac{3}{4}\sqrt{\frac{5}{2\pi}}\,yz(7z^2-3)$

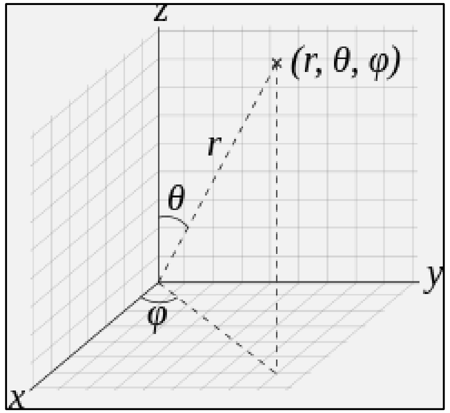

$Y_4^{0}(x,y,z) = \frac{3}{16}\sqrt{\frac{1}{\pi}}\,(35z^4-30z^2+3)$


$Y_4^{1}(x,y,z) = \frac{3}{4}\sqrt{\frac{5}{2\pi}}\,xz(7z^2-3)$


$Y_4^{2}(x,y,z) = \frac{3}{8}\sqrt{\frac{5}{\pi}}\,(x^2-y^2)(7z^2-1)$


$Y_4^{3}(x,y,z) = \frac{3}{4}\sqrt{\frac{35}{2\pi}}\,(x^2-3y^2)xz$


$Y_4^{4}(x,y,z) = \frac{3}{16}\sqrt{\frac{35}{\pi}}\,\left(x^2(x^2-3y^2)-y^2(3x^2-y^2)\right)$



$x = \sin\theta\cos\phi$
$y = \sin\theta\sin\phi$
$z = \cos\theta$

$$\hat{R}_{\mathrm{Z}}(\alpha)_{i,j} = \int_0^{2\pi}\int_0^{\pi} Y_4^i(\sin\theta\cos(\phi+\alpha), \sin\theta\sin(\phi+\alpha), \cos\theta)\cdot Y_4^j(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\sin\theta\, d\theta\, d\phi$$

$$\hat{R}_{\mathrm{X}}\left(\frac{\pi}{2}\right)_{i,j} = \int_0^{2\pi}\int_0^{\pi} Y_4^i(\sin\theta\cos\phi, -\cos\theta, \sin\theta\sin\phi)\cdot Y_4^j(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\sin\theta\, d\theta\, d\phi$$

➔ use Computer Algebra Systems (CAS): Mathematica, Maple, Sage

Sage code (https://cloud.sagemath.com)

```
Y4_4(x,y,z) = 3/4*sqrt(35/pi)*x*y*(x^2 - y^2)
Y4_3(x,y,z) = 3/4*sqrt(35/(2*pi))*(3*x^2 - y^2)*y*z
Y4_2(x,y,z) = 3/4*sqrt(5/pi)*x*y*(7*z^2 - 1)
Y4_1(x,y,z) = 3/4*sqrt(5/(2*pi))*y*z*(7*z^2 - 3)
Y40(x,y,z) = 3/16*sqrt(1/pi)*(35*z^4 - 30*z^2 + 3)
Y41(x,y,z) = 3/4*sqrt(5/(2*pi))*x*z*(7*z^2 - 3)
Y42(x,y,z) = 3/8*sqrt(5/pi)*(x^2 - y^2)*(7*z^2 - 1)
Y43(x,y,z) = 3/4*sqrt(35/(2*pi))*(x^2 - 3*y^2)*x*z
Y44(x,y,z) = 3/16*sqrt(35/pi)*(x^2*(x^2 - 3*y^2) - y^2*(3*x^2 - y^2))

Y4 = [Y4_4, Y4_3, Y4_2, Y4_1, Y40, Y41, Y42, Y43, Y44]
```

```
for i in range(9):
    v = []
    for j in range(9):
        Si(theta, phi) = Y4[i](sin(theta)*cos(phi+a), sin(theta)*sin(phi+a), cos(theta))
        Sj(theta, phi) = Y4[j](sin(theta)*cos(phi), sin(theta)*sin(phi), cos(theta))
        v.append(integral(Si(theta, phi) * Sj(theta, phi) * sin(theta), theta, 0, pi).integrate(phi, 0, 2*pi))
    print v
```

```
for i in range(9):
    v = []
    for j in range(9):
        Si(theta, phi) = Y4[i](sin(theta)*cos(phi), -cos(theta), sin(theta)*sin(phi))
        Sj(theta, phi) = Y4[j](sin(theta)*cos(phi), sin(theta)*sin(phi), cos(theta))
        v.append(integral(Si(theta, phi) * Sj(theta, phi) * sin(theta), theta, 0, pi).integrate(phi, 0, 2*pi))
    print v
```

Matrices written down

$$\hat{R}_{\mathrm{Z}}(\alpha) = \begin{pmatrix} \cos 4\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin 4\alpha \\ 0 & \cos 3\alpha & 0 & 0 & 0 & 0 & 0 & \sin 3\alpha & 0 \\ 0 & 0 & \cos 2\alpha & 0 & 0 & 0 & \sin 2\alpha & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & 0 & \sin\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin\alpha & 0 & \cos\alpha & 0 & 0 & 0 \\ 0 & 0 & -\sin 2\alpha & 0 & 0 & 0 & \cos 2\alpha & 0 & 0 \\ 0 & -\sin 3\alpha & 0 & 0 & 0 & 0 & 0 & \cos 3\alpha & 0 \\ -\sin 4\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos 4\alpha \end{pmatrix}$$

$$\hat{R}_{\mathrm{X}}\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \sqrt{14}/4 & 0 & -\sqrt{2}/4 & 0 \\ 0 & -3/4 & 0 & \sqrt{7}/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}/4 & 0 & \sqrt{14}/4 & 0 \\ 0 & \sqrt{7}/4 & 0 & 3/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3/8 & 0 & \sqrt{5}/4 & 0 & \sqrt{35}/8 \\ -\sqrt{14}/4 & 0 & -\sqrt{2}/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{5}/4 & 0 & 1/2 & 0 & -\sqrt{7}/4 \\ \sqrt{2}/4 & 0 & -\sqrt{14}/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{35}/8 & 0 & -\sqrt{7}/4 & 0 & 1/8 \end{pmatrix}$$

Going between ZYZ (3D) ⇔ SH (9D)

- ZYZ ➔ SH $\quad \mathbf{f}(\alpha,\beta,\gamma) \coloneqq \hat{R}_Z(\gamma)\cdot\hat{R}_X\left(-\frac{\pi}{2}\right)\cdot\hat{R}_Z(\beta)\cdot\hat{R}_X\left(\frac{\pi}{2}\right)\cdot\hat{R}_Z(\alpha)\cdot\hat{h}$

$$\hat{h} \coloneqq \begin{pmatrix} 0\\0\\0\\0\\\sqrt{7}\\0\\0\\0\\\sqrt{5} \end{pmatrix}$$

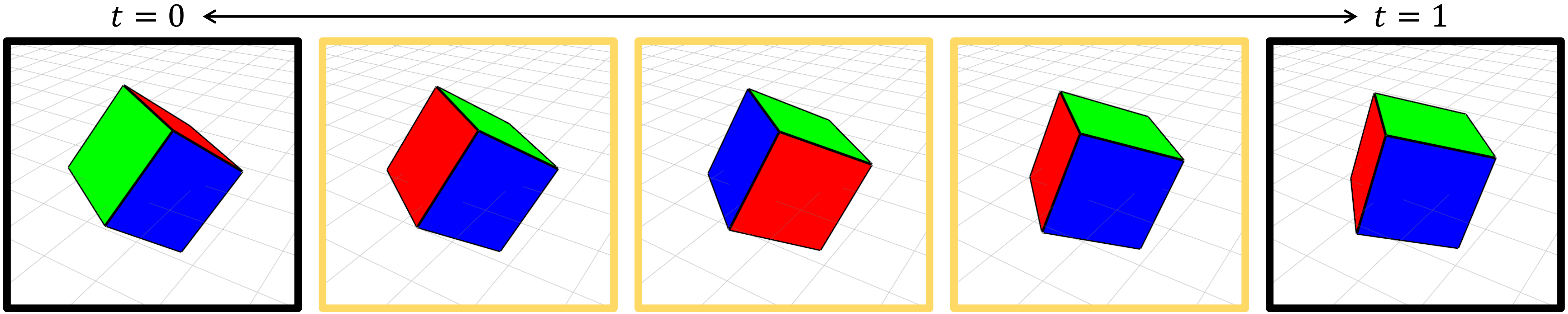
- SH ➔ ZYZ $\quad \mathbf{g}(\mathbf{f}_{\text{in}}) \coloneqq \underset{(\alpha,\beta,\gamma)\in\mathbb{R}^3}{arg\,\min} \|\mathbf{f}_{\text{in}} - \mathbf{f}(\alpha,\beta,\gamma)\|^2$
  - Minimize $E(\alpha,\beta,\gamma)$ using Conjugate Gradient

$$\frac{dE}{d\alpha} = -2\left(\mathbf{f}_{\text{in}} - \mathbf{f}(\alpha,\beta,\gamma)\right)^{\top}\frac{d\mathbf{f}}{d\alpha}$$

$$\frac{d\mathbf{f}}{d\alpha} = \hat{R}_Z(\gamma)\cdot\hat{R}_X\left(-\frac{\pi}{2}\right)\cdot\hat{R}_Z(\beta)\cdot\hat{R}_X\left(\frac{\pi}{2}\right)\cdot\frac{d\hat{R}_Z}{d\alpha}\cdot\hat{h}$$

$$\frac{d\hat{R}_Z}{d\alpha} = \begin{pmatrix}
-4\sin 4\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4\cos 4\alpha\\
0 & -3\sin 3\alpha & 0 & 0 & 0 & 0 & 0 & 3\cos 3\alpha & 0\\
0 & 0 & -2\sin 2\alpha & 0 & 0 & 0 & 2\cos 2\alpha & 0 & 0\\
0 & 0 & 0 & -\sin\alpha & 0 & \cos\alpha & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\cos\alpha & 0 & -\sin\alpha & 0 & 0 & 0\\
0 & 0 & -2\cos 2\alpha & 0 & 0 & 0 & -2\sin 2\alpha & 0 & 0\\
0 & -3\cos 3\alpha & 0 & 0 & 0 & 0 & 0 & -3\sin 3\alpha & 0\\
-4\cos 4\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4\sin 4\alpha
\end{pmatrix}$$

Toy example: interpolating two frames

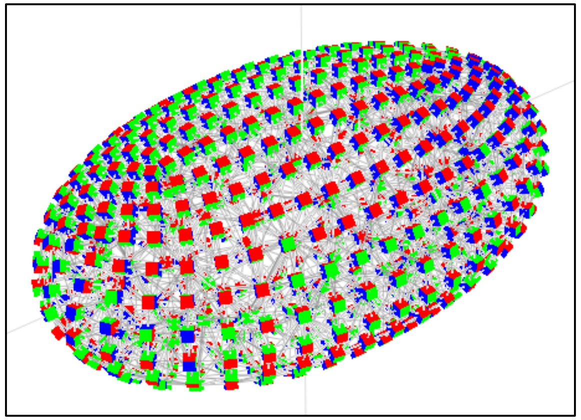


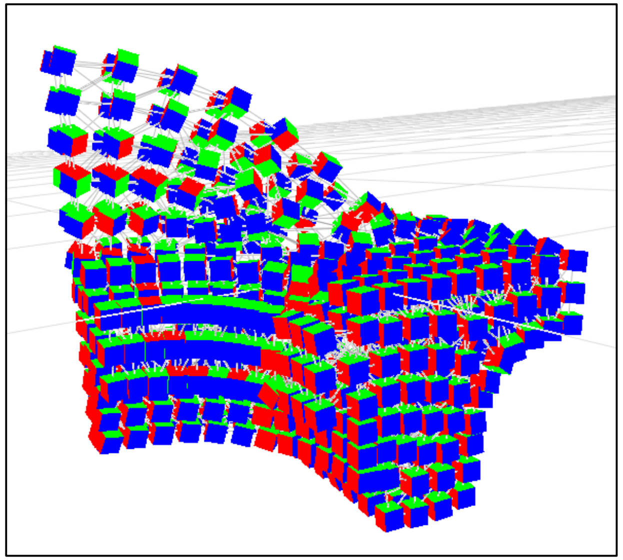
$\mathbf{f}_0 = \mathbf{f}(\alpha_0, \beta_0, \gamma_0)$
$\alpha_0 = 53°$
$\beta_0 = 60°$
$\gamma_0 = 356°$

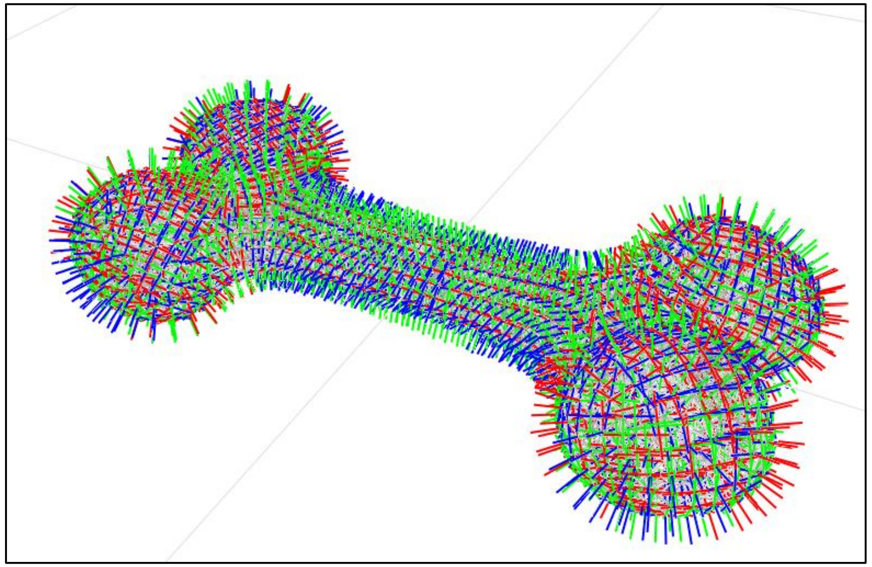
$$(\alpha_t, \beta_t, \gamma_t) = \underset{(\alpha,\beta,\gamma)\in\mathbb{R}^3}{arg\min} \|(1-t)\,\mathbf{f}_0 + t\,\mathbf{f}_1 - \mathbf{f}(\alpha,\beta,\gamma)\|^2$$

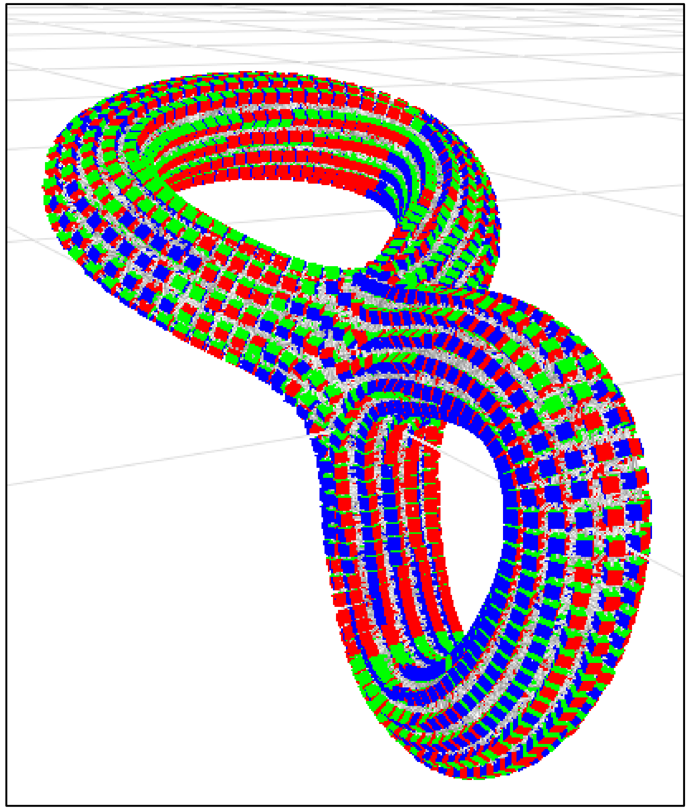
$\mathbf{f}_1 = \mathbf{f}(\alpha_1, \beta_1, \gamma_1)$
$\alpha_1 = 160°$
$\beta_1 = 43°$
$\gamma_1 = 2°$

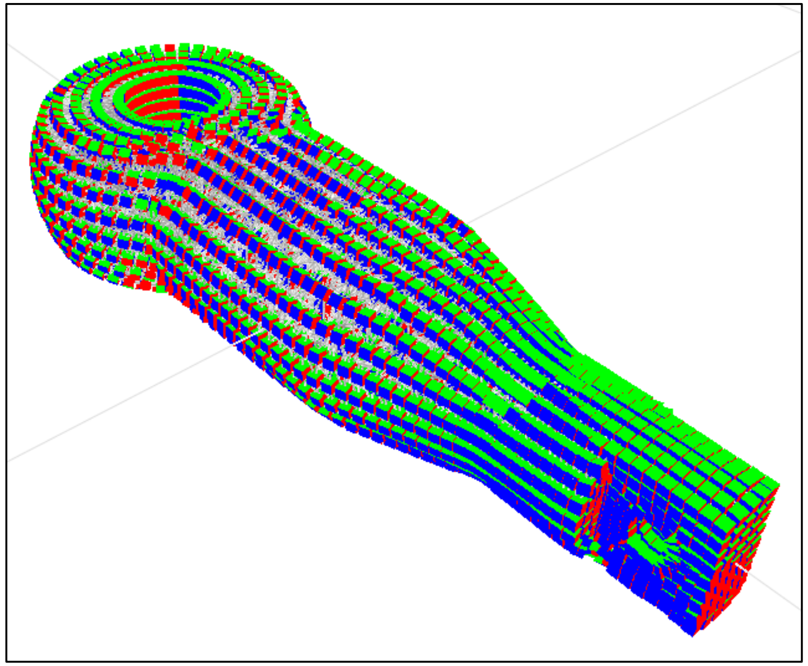
Real examples with tetrahedral meshes

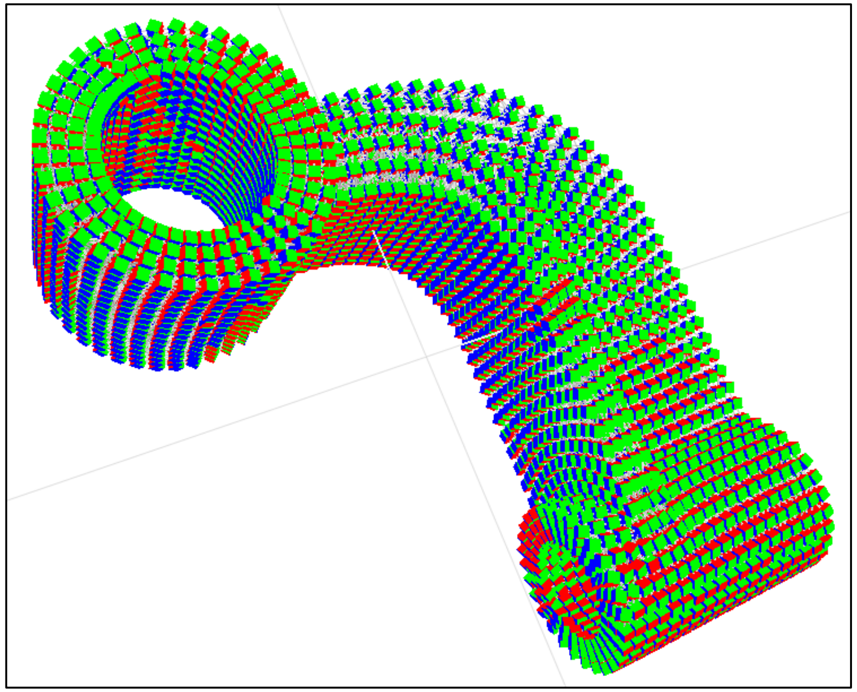












Small differences from [Huang11]

- Per-vertex discretization
  - ⬄ per-face (Crouzeix-Raviart) discretization
  - #vertices $\ll$ #faces ➔ much smaller problem size (x0.1)
  - Normal alignment properly handled

- No global nonlinear solve w.r.t. $\{\ (\alpha_i, \beta_i, \gamma_i)\ \}$
  - Only Laplacian smoothing + per-vertex nonlinear projection

Recent arXiv paper [Ray & Sokolov 2015]

- Unified view toward 2D & 3D problems
- Better handling of normal alignment
- “Feasibility” constraint linearized & integrated into iterative solve
- SH cookbook, concise pseudocode

On Smooth Frame Field Design

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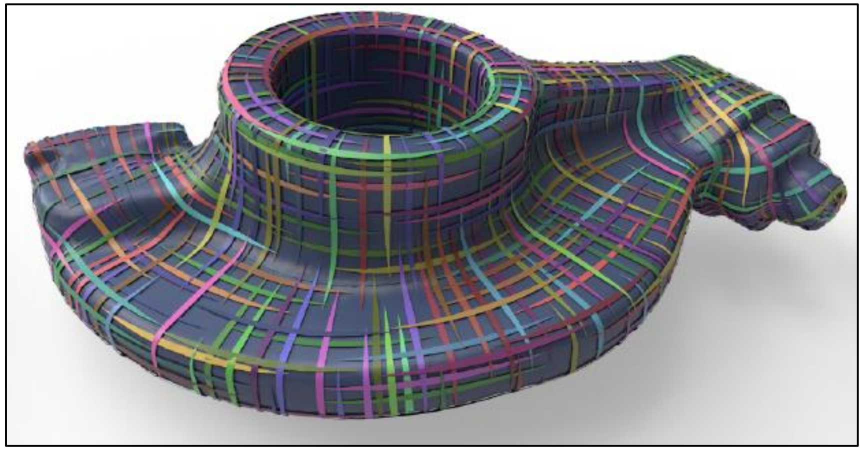
arXiv:1507.03351v1 [cs.GR] 13 Jul 2015

1. INTRODUCTION

The 2D algorithm with frames represented by functions

http://arxiv.org/abs/1507.03351

Small ideas for further improvements



- Fix 3D frames on boundary using 2D frames
  - Decouple 9 SH coeffs ➔ x1/9 dimensionality
- Jacobi-style iteration ➔ simple & parallel
- Other scattered-data-interpolation tools (RBF / MLS) ?
- (Not my main focus anyway)

Instant Field-Aligned Meshes [Jakob, Tarini, Panozzo, Sorkine-Hornung, SIGGRAPH Asia 2015]